JOURNAL OF PURE AND APPLIED ALGEBRA

ELSEVIER Journal of Pure and Applied Algebra 110 (1996) 15–56

Artin groups of extra-large type are biautomatic

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Communicated by C.A. Weibel; received 15 August 1994; revised 10 April 1995

Abstract

In this paper we develop new techniques to work with small cancellation theory diagrams for Artin groups. Using these techniques we examine paths in the Cayley graph of the Artin group. For any Artin group G, with semigroup generators $\mathscr{A}$, we define a language $L(G) \subset \mathscr{A}^*$. The language $L(G)$ is a set of canonical forms for the Artin group. In the case G is an Artin group of extra-large type or a two generator Artin group, we analyze the geometry of the small cancellation theory diagrams and show that $L(G)$ is the language of a biautomatic structure for G.

0. Introduction

The main objective of this paper is to exhibit biautomatic structures for many of the Artin groups of infinite type. The class of biautomatic groups was first introduced by the authors of [6]. Intuitively, this is the class of groups for which there is a set of finite state algorithms containing instructions for constructing any bounded portion of the Cayley graph for the group. Many classes of groups are known to have a biautomatic structure, among these are finitely generated abelian groups, hyperbolic groups and small cancellation groups. For more background on biautomatic groups see [6, 3, 7, 8].

Artin groups were first defined by Brieskorn as a generalization of the braid groups. The Artin groups have standard presentation $\mathscr{P} = \langle \mathscr{X} \mid \mathscr{R} \rangle$ defined as follows. The set of generators $\mathscr{X}$ is a finite set $\{x_i \mid i \in I\}$. To each pair $i \neq j \in I$ there is associated a value $m_{ij} \in \mathbb{N} \cup \infty$, such that each $m_{ij} > 1$ and $m_{ij} = m_{ji}$. Let $F(\mathscr{X})$ denote the free group on $\mathscr{X}$. For each finite m_{ij} the word $h_{ij} \in F(\mathscr{X})$ is defined to be the string of alternating generators x_i and x_j of length m_{ij} starting with x_i. For each finite m_{ij}, define $r_{ij} = h_{ij} h_{ji}^{-1}$, where h_{ji}^{-1} is the inverse of the word h_{ji} in $F(\mathscr{X})$. The set of relators for the

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presentation $\mathscr{P}$ is

$$\mathscr{R} = \{r_{ij} | m_{ij} \neq \infty\}.$$

In this paper Artin groups will be denoted by H and are assumed to be given by a presentation $\mathscr{P} = \langle \mathscr{X} | \mathscr{R} \rangle$ as defined.

Associated to each Artin group is a Coxeter group, C_H. The Coxeter group is the epimorphic image of the Artin group obtained by expanding the set of relators $\mathscr{R}$ to include x_i^2 for each $i \in I$. An Artin group H is said to be of *finite type* if C_H is finite, otherwise it is said to be of *infinite type*. For the braid group on n strings C_H is the symmetric group S_n. Consequently, the braid groups are Artin groups of finite type.

W. Thurston has shown that the braid groups are biautomatic [6]. This result has been extended by Charney [4] to show that all Artin groups of finite type are biautomatic. The goal in this paper is to show that many of the Artin groups of infinite type are also biautomatic. In [2], Appel and Schupp define two subclasses of the Artin groups of infinite type. An Artin group is said to be of large type if all $m_{ij} > 2$, and to be of extra-large type if all $m_{ij} > 3$. Notice that extra-large type is a subclass of large type which in turn is a subclass of infinite type. The main result of this paper is:

Theorem A. *Artin groups of extra-large type are biautomatic.*

Suppose that H is an Artin group of extra-large type with presentation $\mathscr{P} = \langle \mathscr{X} | \mathscr{R} \rangle$. The set $\mathscr{A} = \mathscr{X} \cup \mathscr{X}^{-1}$ is called the set of semigroup generators of H and $\mathscr{A}^*$ is the free monoid on $\mathscr{A}$. To prove Theorem A we use a characterization of biautomatic given in [6]. This states that H is biautomatic if there is a group language $L(H) \subset \mathscr{A}^*$ that is both regular and a bicombing of the Cayley graph $\Gamma_{\mathscr{A}}(H)$. This characterization is explained in Section 1.

In Section 2 we construct a regular ordering $\prec$ of $\mathscr{A}^*$ and use the ordering to define a group language $L(H) \subset \mathscr{A}^*$. To show that $L(H)$ is regular we use a lemma. Let $\overline{L(H)}$ denote the compliment of $L(H)$ in $\mathscr{A}^*$. Intuitively, the lemma states that a language defined as we have done, by a regular ordering, is regular if each word $\omega \in \overline{L(H)}$ is close to a word $v \in \mathscr{A}^*$ with the properties, $v = \omega$ in H, and $v \prec \omega$. Two words are close if their corresponding paths in the $\Gamma_{\mathscr{A}}(H)$ are fellow travelers. This is explained in Section 1. The word v, with the properties given above, is said to refute the word ω. Thus, $L(H)$ is regular if each word ω in $\overline{L(H)}$ is refuted.

In order to compare paths in $\Gamma_{\mathscr{A}}(H)$ we use equality diagrams from small cancellation theory. Artin groups of extra-large type do not in general satisfy small cancellation hypotheses. However, small cancellation techniques can be used to analyze these groups. Both Sections 3 and 4 briefly review the necessary background concerning the study of Artin groups via small cancellation theory. In Section 5, we define thin equality diagrams and exhibit some relations with the geometry of $\Gamma_{\mathscr{A}}(H)$. In Section 6 $C(4)$–$T(4)$ equality diagrams are examined. The bulk of the proof of Theorem A is contained in Sections 7–9, here it is shown that all words in $\overline{L(H)}$ are refuted. This is done in stages by examining words which could label the boundary of more and more complicated equality diagrams. Finally, in Section 10 we prove that $L(H)$ is regular and a bicombing of $\Gamma_{\mathscr{A}}(H)$.

1. Definitions

In this paper we will give only the essential definitions. The reader can refer to [6, 11] for more details concerning automatic groups and small cancellation theory, respectively.

Suppose that G is a finitely generated group given by the presentation $\langle \mathscr{X} | \mathscr{R} \rangle$. We will always assume that the set of generators $\mathscr{X}$ is finite. Let $\mathscr{A} = \mathscr{X} \cup \mathscr{X}^{-1}$ be the set of semigroup generators. Let $F(\mathscr{X})$ be the free group on the set $\mathscr{X}$, and let $\mathscr{A}^*$ be the free monoid on the set $\mathscr{A}$. We can think of $\mathscr{A}$ as an alphabet and $\mathscr{A}^*$ as the language of all finite words over $\mathscr{A}$. In this context, $F(\mathscr{X}) \subset \mathscr{A}^*$ is the language of all words over $\mathscr{A}$ which are freely reduced. For any element $\omega \in \mathscr{A}^*$ define $\bar{\omega} \in G$ to be the image of ω in the group G under the endomorphism $\mathscr{A}^* \to G$ defined by the presentation.

The Cayley graph for G with respect to the set $\mathscr{A}$, denoted by $\Gamma_{\mathscr{A}}(G)$, is a directed labeled graph. The vertex set is G and there is an edge labeled by a from g to ga for each $a \in \mathscr{A}$ and $g \in G$. A metric is defined on $\Gamma_{\mathscr{A}}(G)$ by considering each edge as isometric to the unit interval. The distance between points x and y is denoted by $d(x, y)$. With this metric $\Gamma_{\mathscr{A}}(G)$ is a geodesic metric space.

The length of a word ω in $\mathscr{A}^*$ is denoted by $|\omega|$. For each word $\omega \in \mathscr{A}^*$ there is a unique edge path in the Cayley graph from the identity to $\bar{\omega}$ which travels at unit speed along the edges labeled by the word ω. We will refer to paths in $\Gamma_{\mathscr{A}}(G)$ by their label. This should not cause any problems, the context will make the distinction clear. For notational convenience we define $\omega(t) = \bar{\omega}$ for $t \geq |\omega|$. So that ω maps the infinite ray $[0, \infty]$ to $\Gamma_{\mathscr{A}}(G)$.

Suppose that $\omega(t)$ and $v(t)$ are two paths in the Cayley graph. The paths are said to be *k-fellow travelers* if for all $t \geq 0$,

$$d(\omega(t), v(t)) \leq k.$$

Another measure of the distance between the paths is given by Hausdorff neighborhoods. The path ω is said to lie in a *k-Hausdorff neighborhood* of the path v if for all $t \geq 0$ there is a number s such that

$$d(\omega(t), v(s)) \leq k.$$

A word in $\mathscr{A}^*$ is called a *geodesic* if it is shortest among all words representing the same group element. For each geodesic word the corresponding path is a geodesic in $\Gamma_{\mathscr{A}}(G)$. The following proposition can be found in [6].

Proposition 1. *Let ω and v be two geodesic paths, in a geodesic metric space, which start a distance l apart. If ω and v lie in a k-Hausdorff neighborhood of each other then ω and v are $(2k+l)$-fellow travelers. Conversely, if ω and v are k-fellow travelers, then the two paths lie in a k-Hausdorff neighborhood of each other.*

A language $L \subset \mathscr{A}^*$ is called a *normal form* of G if the natural map $L \to G$ is surjective. This implies that for any group element $g \in G$ there is at least one word in L representing a path in $\Gamma_{\mathscr{A}}(G)$ from the origin to the vertex g. A normal form $L \subset \mathscr{A}^*$ is called a *bicombing* of G if there exist a constant k satisfying the following property. Given $a, b \in \mathscr{A} \cup \emptyset$ and $\omega, v \in L$ with $\overline{\omega a} = \overline{bv}$ in G then the path ω, starting at the origin in $\Gamma_{\mathscr{A}}(G)$, and the path v, starting at the vertex $\bar{b}$ in $\Gamma_{\mathscr{A}}(G)$, are k-fellow travelers. (In other recent papers, some authors may refer to this type of bicombing as a bounded bicombing).

A language is said to be *regular* if it is accepted by a finite state automaton. For more details on finite state automata and regular languages see [6] or [10]. The standard definition of a biautomatic group is based on a set of finite state automata which are used to compare words represent near by group elements. We will use a more geometric characterization given by the following proposition from [6].

Proposition 2. *Let G be a group with semigroup generators $\mathscr{A}$. The group G is biautomatic if there is a regular language $L \subset \mathscr{A}^*$ that is a bicombing of $\Gamma_{\mathscr{A}}(G)$. In this case we say that L is the language of a biautomatic structure for G.*

2. The language for Artin groups

Let $H = \langle \mathscr{X} | \mathscr{R} \rangle$ be an Artin group. Let $\mathscr{A} = \mathscr{X} \cup \mathscr{X}^{-1}$ be the set of semigroup generators. In this section we define a language $L(H)$ which is a subset of $\mathscr{A}^*$. In the rest of the paper we will show that $L(H)$ is a regular language and a bicombing of H, for two families of Artin groups: two generator Artin groups, and Artin groups of extra-large type. By Proposition 2, this will show that for these Artin groups $L(H)$ is the language of a biautomatic structure for the group H.

Any word ω in $\mathscr{A}^*$ can be written uniquely as a product of powers of semigroup generators in the form $a_1^{n_1} a_2^{n_2} \ldots a_l^{n_l}$ where each $a_i \in \mathscr{A}$, $n_i \in \mathbb{N}$, and no consecutive pair of a_i's are equal. We call the subwords $a_i^{n_i}$ the *syllables* of the word ω. Define $\|\omega\|$ to be the number of syllables in the word ω.

Assign to each generator in $\mathscr{X}$ a distinct *color*. Assign to each inverse generator $x_i^{-1} \in \mathscr{X}^{-1}$ the same color as the generator $x_i \in \mathscr{X}$. Because syllables are powers of a single semigroup generator, each syllable of a word has assigned colors.

To each word $\omega \in \mathscr{A}^*$ we assign a string of 0's and 1's. This binary string is denoted by $\mathscr{B}(\omega)$. There is exactly one digit in $\mathscr{B}(\omega)$ for each letter in ω. The string $\mathscr{B}(\omega)$ is calculated by mapping each letter of ω to either 0 or 1, as defined below.

Definition ($\mathscr{B}(\omega)$, *the binary string associated with* ω). Each letter of ω is mapped to $\{0,1\}$ by the following procedure. All letters that are not the last letter of a syllable are mapped to 1. The last letter of the first and last syllable are mapped to 0. Suppose a_n is the nth letter of ω and the last letter of syllable k. Let $\mathscr{CO}(k)$ be the color of the kth syllable. The letter a_n is mapped to

0 if $\mathscr{CO}(k-1) = \mathscr{CO}(k+1)$ and

1 otherwise.

The binary number $\mathscr{B}(\omega)$ is the number whose highest digit corresponds to the first letter of ω, second highest digit corresponds to the second letter of ω, and so on.

Choose an ordering of the elements of $\mathscr{A}$. This induces a lexicographic ordering, $\prec_l$, of $\mathscr{A}^*$. The ordering used to define the bicombing of the Artin groups is created from the word length, the binary numbers, and the lexicographic ordering.

Definition ($\prec$, *a total ordering of* $\mathscr{A}^*$). Given two words $\omega, v \in \mathscr{A}^*$ we say that ω precedes the word v, denoted by $\omega \prec v$, if
(1) $|\omega| < |v|$ or,
(2) $|\omega| = |v|$ and $\mathscr{B}(\omega) < \mathscr{B}(v)$ or,
(3) $|\omega| = |v|$, $\mathscr{B}(\omega) = \mathscr{B}(v)$ and $\omega \prec_l v$.

An ordering $\prec$ is said to be *regular* if the language $L_\prec = \{(v,\omega) \mid v \prec \omega\}$ is regular. For our purposes it is safe to think of $L_\prec$ as a sublanguage of $\mathscr{A}^* \times \mathscr{A}^*$. But in fact, a more rigorous approach would consider the embedding of $\mathscr{A}^* \times \mathscr{A}^*$ in the padded language associated with $\mathscr{A}^*$. For simplicity, we will avoid this technicality and refer the reader [6, 3] for information on padded languages.

Lemma 3. *The ordering* $\prec$ is regular.

Proof. Suppose that M is a machine reading a word of $\mathscr{A}^*$. To determine the digit corresponding to a given letter, a, the machine only needs to know the color of the last syllable and the color of the next letter. If the next letter has the same color as a then a is not the end of a syllable and is therefore mapped to 1. If the next letter has a different color than a then the machine can compare the colors of the last syllable and the next letter to determine whether a is mapped to 1 or 0. Therefore, the machine would need only a finite memory. Therefore, we can construct a finite state automaton that can compare the binary strings of two words of $\mathscr{A}^*$, and accepts the language $L_{\mathscr{B}<} = \{(\omega, v) \mid \mathscr{B}(\omega) < \mathscr{B}(v)\}$. We can also construct an automaton which accepts the language $L_{\mathscr{B}=} = \{(\omega, v) \mid \mathscr{B}(\omega) = \mathscr{B}(v)\}$. Therefore, $L_{\mathscr{B}<}$ and $L_{\mathscr{B}=}$ are regular languages. The three languages $L_< = \{(\omega, v) \mid |\omega| < |v|$, $L_= = \{(\omega, v) \mid |\omega| = |v|\}$, and $L_{\text{lex}<} = \{(\omega, v) \mid \omega \prec_l v\}$, are regular by standard result in language theory. Regular languages are closed under the Boolean operations of intersection and union.

Therefore, the language

$$L_{\prec} = L_{<} \cup (L_{=} \cap L_{\mathscr{B}<}) \cup (L_{=} \cap L_{\mathscr{B}=} \cap L_{\text{lex}<})$$

is regular. □

Definition (*The language $L(H)$*). The language $L(H) \subset \mathscr{A}^*$ consists of all $\prec$ minimal representatives for each group element

$$L(H) = \{\omega \in \mathscr{A}^* \mid \omega \text{ is } \prec \text{ minimal for } \bar{\omega}\}.$$

Because the lexicographic ordering is total the language $L(H)$ will have exactly one representative for each group element. Thus, $L(H)$ is a set of canonical forms for the Artin group H. Furthermore, note that each word in $L(H)$ is shortest among all words in $\mathscr{A}^*$ representing the same group element. Thus, non-geodesic words do not occur as subwords of elements of $L(H)$. There is an equally important set of non-admissible subwords. These words are called the *excessive words* of H. They do not appear as subwords because there are equivalent words of equal length which precede them in the ordering $\prec$. Before defining the set of excessive words it will help to consider the motivating example, far-corners.

Let r_{ij} be a relator of H. Let γ be a subword of a cyclic permutation of r_{ij}, with length m_{ij}. We call γ a half relator of H. Suppose that γ begins with the semigroup generator a. The word $a\gamma$ is called a *type one far-corner*. Let b be the second letter of γ and let c be any other letter in $\mathscr{A}$ with color distinct from both a and b. Words of the form

$$bc^n\gamma \quad bC^n\gamma \quad Bc^n\gamma \quad BC^n\gamma.$$

are called *type two far-corners*. (The notation C denotes the other semigroup generator of the same color as c.) The next lemma shows that these words never occur as subwords in elements of $L(H)$.

Lemma 4. *Let H be any Artin group. Then far-corners will not appear as subwords of an element of $L(H)$.*

Proof. Suppose that ω contains a type one far-corner subword. Thus, ω contains a subword $a\gamma$, with γ a half relator starting with a. First examine the case when γ is a half relator of letters all with the same sign as a. Suppose that b is the second letter in the half relator γ. Thus, γ has the form $aba\ldots$ of finite length corresponding to the appropriate m_{ij} value.

By the form of the relator there is another half relator word starting with b which represents the same group element as γ. Call this word γ'. It will have the form $bab\ldots$ Let v be the word obtained from ω by replacing the occurrence of γ in ω by γ'. Let α and ζ be words in $\mathscr{A}^*$ such that

$$\omega = \alpha a(aba\ldots)\zeta \quad \text{and} \quad v = \alpha a(bab\ldots)\zeta.$$

In the strings $\mathscr{B}(\omega)$ and $\mathscr{B}(v)$ the digits associated with the letters of α will be the same because these values only depend on the initial subword αa which is the same in both ω and v. We will concentrate on the digits of $\mathscr{B}(\omega)$ and $\mathscr{B}(v)$ which are associated with the first two letters of the far-corner. First consider the case that the second to last syllable of αa has the same color as b. In this case the letters aa in ω go to the digits 1, 0, but the corresponding letters ab of the word v go to 0, 0:

$$\omega = \dots aa \dots \quad \text{and} \quad \mathscr{B}(\omega) = \dots 10 \dots$$

$$v = \dots ab \dots \quad \text{and} \quad \mathscr{B}(v) = \dots 00 \dots$$

The words ω and v both have the same length and $\mathscr{B}(v) < \mathscr{B}(\omega)$, therefore $v \prec \omega$. The same argument holds if α is the empty word or is a single syllable of the same color as a.

When the next to last syllable of αa is not the same color as b, the letters aa in ω go to the digits 1, 1, but the corresponding letters ab of the word v go to 1, 0:

$$\omega = \dots aa \dots \quad \text{and} \quad \mathscr{B}(\omega) = \dots 11 \dots$$

$$v = \dots ab \dots \quad \text{and} \quad \mathscr{B}(v) = \dots 10 \dots$$

Therefore again, $\mathscr{B}(v) < \mathscr{B}(\omega)$ and because the words have the same length, $v \prec \omega$.

The argument above is based on the ordering of $\mathscr{A}^*$ and the ordering depends only on the color of letters and not on their parity. Therefore, the proof will hold for γ any half relator. In particular, we did not need to assume that the letters of γ all had the same sign as a.

Now consider the type two far-corners. Let ω be a word with a type two far-corner subword, f. For notational purposes, suppose that f has the form, $bc^n(aba \dots)$. Let v be the word which is obtained by replacing the occurrence of the half relator, $aba \dots$, in the word ω by the corresponding half relator starting with b. Let α and ζ be the subwords of ω and v such that

$$\omega = \alpha bc^n(aba \dots)\zeta \quad \text{and} \quad v = \alpha bc^n(bab \dots)\zeta.$$

All the letters of ω and v, up to but not including the last c of the far-corner, are mapped to the same digit in both strings. Consider the last c in the far-corner. In ω it is mapped to a 1, but in the word v it is mapped to a 0. Therefore, since the words are the same lengths we have $v \prec \omega$.

Note again that we did not make essential use of the notation and the proof generalizes to all type two far-corners. □

The more general class of excessive words have all the essential features of far-corner words needed in the proof above. Let a, b and c be letters of $\mathscr{A}$ each having distinct colors. Let A, B, and C be the semigroup generators corresponding to the inverses of a, b and c. Let γ and γ' be any words in $\{a, A, b, B\}^*$ such that

(1) $\bar{\gamma} = \bar{\gamma}'$ in the group H,

(2) $|\gamma| = |\gamma'|$,

(3) γ starts with a,
(4) γ' begins with ba, or bA.
Then the following words are called *excessive words*:

$$a\gamma \quad bc^n\gamma \quad bC^n\gamma \quad Bc^n\gamma \quad BC^n\gamma.$$

Just as in the case for the far-corners we can replacing the subword γ with γ' in the excessive word. This creates a new word which represents the same group element but is lower in the ordering. Therefore, we have the following lemma.

Lemma 5. *Let H by any Artin group. Excessive-words will never appear as subwords in elements of L(H).*

The words γ and γ' play a crucial role in proving Theorem A. To avoid confusion we assign them names. The word γ is called the *high word* and the word γ' is called the *low word*. In a far-corner the high word and the low word are the half relators. Lemma 5 can now be restated as:

Corollary 6. *If ω contains an excessive subword and v is the word obtained by replacing the associated high word with the associated low word then $v \prec \omega$ and $\bar{\omega} = \bar{v}$.*

3. Small cancellation theory and $\mathcal{R}$-diagrams

The material in this section is meant to highlight the concepts from small cancellation theory that will be used in the rest of this paper. For more details the reader can refer to [11, 2]. We will use the standard notation and terminology that is used in these two works.

Let G be a group with presentation $\langle \mathcal{X} | \mathcal{R} \rangle$. The set of relators is said to be *symmetrized* if each relator is cyclically reduced and the set is closed under cyclic permutation and inversion. When using techniques of small cancellation theory, we assume that all sets of relators are symmetrized. An *$\mathcal{R}$-diagram M* is a diagram in the plane consisting of regions, edges, and vertices. Every region is homeomorphic to a disk and is bounded by edges. Every edge in M is labeled by a word from the free group on $\mathcal{X}$, and the label reading around the boundary edges of any region of M is a reduced element of $\mathcal{R}$. Suppose that D_1 and D_2 are two regions in an $\mathcal{R}$-diagram M which share an edge e. The diagram M is *unreduced* if the label reading around D_1 in the clockwise direction, starting at the edge e, and the label reading around D_2 in the counterclockwise direction, starting at the edge e, are the same. The diagram M is said to be *reduced* if this situation never occurs. In an unreduced diagram it is always possible to remove the two regions D_1 and D_2 and sew the remaining edges together to create a new diagram with the original boundary word. Because we are primarily concerned with the boundary words, we can and will assume that all $\mathcal{R}$-diagrams are reduced. A fundamental result of small cancellation theory is:

Proposition 7. *For any word $\omega \in \mathscr{A}^*$ such that $\bar{\omega} = 1$ in the group $G = \langle \mathscr{X} | \mathscr{R} \rangle$ there is a connected, simply connected $\mathscr{R}$-diagram which has boundary labeled by the word ω. Conversely, if there is a connected, simply connected $\mathscr{R}$-diagram with boundary labeled by the word ω, then $\bar{\omega} = 1$ in the group $G = \langle \mathscr{X} | \mathscr{R} \rangle$.*

We will assume that all $\mathscr{R}$-diagrams are connected and simply connected.

In order to discuss $\mathscr{R}$-diagrams we need some notation and terminology. The following is the standard notation and terminology as appearing in [11, 2]. Suppose M is an $\mathscr{R}$-diagram with vertex v, edge e, and region D. The *boundary* of M is denoted by ∂M. The *boundary* of D is denoted by ∂D. The vertex v is called a *boundary vertex* if $v \in \partial M$; otherwise, v is called an *interior vertex*. The *degree* of v, $d(v)$, is the number of edges incident to v counting multiplicity. The edge e is called a *boundary edge* if $e \in \partial M$; otherwise, e is called an *interior edge*. The region D is called a *boundary region* of M if $\partial D \cap \partial M \neq \emptyset$; otherwise, D is called an *interior region*. A boundary region D is called *almost interior* if ∂D contains no boundary edges. The boundary region D is called a *simple boundary region* if $\partial D \cap \partial M$ is a consecutive portion of ∂M, and a *non-simple boundary region* otherwise. The *degree* of D, $d(D)$, is the number of edges, counting multiplicity, in any boundary cycle of D. The *interior degree* of D, $i(D)$, is the number of interior edges, counting multiplicity, in any boundary cycle of D.

A word $p \in F(\mathscr{X})$ is called a *piece* if there exists distinct elements r_1 and r_2 of $\mathscr{R}$ such that p is a prefix of both r_1 and r_2. Pieces are exactly the set of subwords of relators that freely cancel in the product of two non-inverse relators. There are properties of the presentation for a group that limit the geometry of $\mathscr{R}$-diagrams whose boundary word is trivial in the group. Two of these properties are $C(p)$ and $T(q)$. The set of relators $\mathscr{R}$ is said to satisfy:

$C(p)$: if no element $r \in \mathscr{R}$ can be written as a product of fewer than p pieces,
$T(q)$: If given $3 < n < q$ and $r_1, r_2, \ldots, r_n$ elements of $\mathscr{R}$ so that no successive pair $r_i r_{i+1}$ forms an inverse pair (including $r_n r_1$), at least one of the products $r_1 r_2$, $r_2 r_3, \ldots, r_{n-1} r_n$, $r_n r_1$ is freely reduced without cancellation.

If an $\mathscr{R}$-diagram contains an interior vertex of degree two then the two edges can be combined to form a single edge labeled by the product of the two original labels. Therefore we will assume that all interior vertices have degree greater than two. In a reduced $\mathscr{R}$-diagram the interior edges are always labeled by pieces. Another fundamental result from small cancellation theory is:

Proposition 8. *Let M be a reduced $\mathscr{R}$-diagram:*

(1) *The assertion $\mathscr{R}$ satisfies $C(p)$, implies that $i(D) \geq p$ for every almost interior region D of M.*

(2) *The assertion $\mathscr{R}$ satisfies $T(q)$, implies that $d(v) \geq q$ for every interior vertex v of M.*

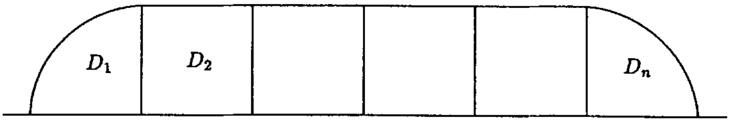

Fig. 1. A compound strip.

We will say that a diagram is $C(p)$-$T(q)$ if it satisfies the conclusions of Proposition 8.

In this paper we are especially interested in $C(4)$–$T(4)$ diagrams. Gersten and Short have shown that all groups that have a $C(4)$–$T(4)$ presentation are biautomatic, see [7, 8]. In general, the presentation for Artin groups does not satisfy $C(4)$–$T(4)$. However, because of the specific form of the relators $\mathscr{R}$ we are able to use small cancellation techniques similar to those used in the $C(4)$–$T(4)$ case. Our investigation focuses on the boundary regions of $\mathscr{R}$-diarams. Therefore, we have some special terminology for boundary regions of $C(4)$–$T(4)$ diagrams. Suppose M is a $C(4)$–$T(4)$ diagram and D is a simple boundary region of M. The region D is called a *singleton strip* if $i(D) = 0$ or 1, a *corner* if $i(D) = 2$, a *side* if $i(D) = 3$, and an *inner corner* if $i(D) = 4$.

We consider sequences of boundary regions which are encountered consecutively when transversing the boundary of M in one of the two directions. A special case of such a sequence is a compound strip. A *compound strip* of M is a sequence of boundary regions $\{D_1, \dots, D_n\}$ encountered consecutively when transversing the boundary of M in one of the two directions such that each D_i is a simple boundary region, the region D_i shares an edge with D_{i+1}, D_1 and D_n are corners, and $D_2, \dots, D_{n-1}$ are sides (see Fig. 1).

One of the most useful tools in small cancellation theory is Lyndon's curvature theorem. This is a combinatorial version of the Gauss–Bonet curvature theorem. For details concerning Lyndon's curvature theorem see [11] or [12]. Instead of stating Lyndon's curvature theorem in its most general form we state three corollaries to the theorem which apply to the specific situations we are concerned with. In the following corollaries summations will be taken over regions and vertices. The notation

$\sum^*$ signifies that the sum is over *simple boundary regions*,
$\sum^\bullet$ signifies that the sum is over *boundary* vertices or *boundary* regions and
$\sum^\circ$ signifies that the sum is over *interior* vertices or *interior* regions.

Corollary 9 (Lyndon and Schupp [11]). *If M is a connected, simply connected diagram satisfying $C(4)$–$T(4)$ which has more than one region, then $\sum^*(3 - i(D)) \geq 4$. If M is a connected, simply connected diagram satisfying $C(6)$ which has more than one region, then $\sum^*(4 - i(D)) \geq 6$.*

A single edge sticking out of a diagram is called a *spike*. Notice that the tip of a spike will be labeled by inverse pair. Therefore, if the boundary word is cyclically reduced then there will not be spikes in the diagram.

Corollary 10 (Appel [1]). *Let M be a connected simply connected $C(4)$–$T(4)$ diagram ith no spikes, no non-simple boundary regions and more than one region. Then*

$$4 = \sum{}^{\bullet}(3 - i(D)) + \sum{}^{\circ}(4 - i(D)) + \sum{}^{\circ}(4 - d(v)).$$

In a diagram two distinct strips can both share a region. This only happens when the shared region is a corner region. Two strips are said to be *disjoint* if there are no shared regions.

Corollary 11 (Appel and Schupp [2]). *Let M be a connected, simply connected $C(4)$–$T(4)$ diagram with no spikes and more than one region. Then one of the following is true:*

(1) *M contains at least two singleton strips.*

(2) *M contains exactly one singleton strip and at least two compound strips.*

(3) *M contains no singleton strips and at least four distinct compound strips.*

In any of the above cases M contains at least two disjoint strips.

We are concerned primarily with paths that lie along the edges and vertices of the $\mathscr{R}$-diagrams. These paths correspond to words in the free monoid $\mathscr{A}^*$. Therefore, when referring to a path through an $\mathscr{R}$-diagram we will always assume the paths lie in the 1-complex of the diagram. We can think of the 1-complex of an $\mathscr{R}$-diagram as a metric space by endowing it with the word metric.

We conclude this section with some terminology for the 1-complex. Suppose that D is a region of an $\mathscr{R}$-diagram M. The term *point* of ∂D refers to any point along ∂D. Thus, vertices are only a few of the points. The *exterior* of D, $ext(D)$, is $\partial D \cap \partial M$. The *open exterior* of D, $openext(D)$, is the topological interior of $ext(D)$. The *base* of D, $base(D)$, is $\partial D - openext(D)$. The *open base* of D, $openbase(D)$, is $\partial D - \mathrm{ext}(D)$. When there is a question as to the diagram concerned, it will be included in the parentheses. For example, the exterior of D with respect to the diagram M is denoted by $ext(D, M)$.

4. Small cancellation theory and Artin groups

Let H be an Artin group given by the presentation $\langle \mathscr{X} | \mathscr{R} \rangle$ described in the introduction. Associated with every Artin group are a number of two generator Artin groups. These groups play a fundamental role in this paper. For each pair of generators (x_i, x_j), denote by H_{ij} the two generator Artin group given by the one relator presentation $\langle x_i, x_j | r_{ij} \rangle$. This group satisfies the condition $C(4)$–$T(4)$. For notational purposes it will be convenient to let $\mathscr{R}_{ij}$ denote the symmetrized set of relators corresponding to r_{ij}. Many of the results in this section concern the two generator Artin groups.

The pieces for the group H_{ij} are the alternating strings of generators of length less than m_{ij} and the alternating strings of inverse generators of length less than m_{ij}. For example the two generator Artin group with presentation

$$\langle x_1, x_2 \mid x_1 x_2 x_1 x_2 x_1^{-1} x_2^{-1} x_1^{-1} x_2^{-1} \rangle$$

has the following set of pieces:

$$\begin{array}{llll} x_1 & x_2 & x_1^{-1} & x_2^{-1} \\ x_1x_2 & x_2x_1 & x_1^{-1}x_2^{-1} & x_2^{-1}x_1^{-1} \\ x_1x_2x_1 & x_2x_1x_2 & x_1^{-1}x_2^{-1}x_1^{-1} & x_2^{-1}x_1^{-1}x_2^{-1}. \end{array}$$

There are two points on the boundary of the region which separate letters of different signs. These points are called the *poles* of the region D. (Appel and Schupp [2] call these *separating vertices.*) An important feature of the poles is that, with respect to the word metric, the poles divide the boundary of the region exactly in half. Each half labeled by a half relator with letters all of the same sign. Because interior edges of an $\mathscr{R}_{ij}$-diagram are always labeled by pieces, poles will never occur within an interior edge, rather only at the vertices. Exhibiting the position of the poles in an $\mathscr{R}$-diagram produces a way to compare lengths of paths.

The following three lemmas state some basic results about the positioning of poles in an $\mathscr{R}_{ij}$-diagram for the two generator Artin groups. The proofs are left to the reader.

Lemma 12. *Let M be an $\mathscr{R}_{ij}$-diagram. Let α be a freely reduced word that labels the boundary of M over the boundary vertex v of degree three. Then the vertex v is a pole for exactly one of the two adjoining regions.*

Lemma 13. *In a reduced $\mathscr{R}_{ij}$-diagram, if two regions share an edge e then neither of the vertices at the ends of e can be poles for both regions.*

Lemma 14. *If v is an interior vertex of degree four in an $\mathscr{R}_{ij}$-diagram then either v is not a pole for any of the four adjoining regions or v is a pole for two non-adjacent regions.*

Let $\mathscr{A}_{ij}$ be the set of semigroup generators for H_{ij}. Recall that the number of syllables in ω is denoted $\|\omega\|$. The next lemma is from [2]. It shows that the word labeling the exterior of a strip is longer then the word labeling the base.

Lemma 15 (Appel and Schupp [2]). *Suppose that Υ is a strip of an $\mathscr{R}_{ij}$-diagram M. Let ω be the label on ext(Υ) and v be the label on base(Υ). Then*

(1) $\|\omega\| \geq m_{ij} + 1$,
(2) $|\omega| \geq |v| + 2$,
(3) $\|\omega\| \geq \|v\| + 2$, *and*
(4) *some region of Υ has both its poles on ext(Υ).*

If $\bar{\omega} = 1$ in H_{ij} then there is a reduced $\mathscr{R}_{ij}$-diagram M for ω. By Corollary 11, either M is one region or M has at least two distinct strips. Since the length of the label on the exterior of a strip exceeds m_{ij} by at least one, we have the following lemma.

Lemma 16. *If $\bar{\omega} = 1$ in H_{ij} then $|\omega| \geq 2m_{ij}$. Furthermore, if $|\omega| = 2m_{ij}$ then any $\mathscr{R}_{ij}$-diagram labeled by ω is a single region and ω is a relator.*

In general, the Artin group $H = \langle \mathscr{X} | \mathscr{R} \rangle$ on more than two generators does not satisfy the condition $C(4)$–$T(4)$. It is however possible that an $\mathscr{R}$-diagram for a word equal to the identity in G may satisfy $C(4)$–$T(4)$ even though the group does not. This is one of the main tricks in this paper. For the boundary words we are concerned with we are able to construct $C(4)$–$T(4)$ diagrams. The next proposition extends the result of Appel and Schupp, stated in Lemma 15, to the general Artin groups.

Proposition 17. *Let $H = \langle \mathscr{X} | \mathscr{R} \rangle$ be any Artin group and let M be a $C(4)$–$T(4)$ reduced $\mathscr{R}$-diagram. Let Υ be a strip of M, let ω be the label on ext(Υ) and v be the label on base(Υ). Then*
(1) $|\omega| \geq |v| + 2$ *and*
(2) $\|\omega\| \geq \|v\| + 2$.

Proof. The strip Υ is a finite sequence of simple boundary regions $\{D_i\}$. Each of the regions D_i is labeled by one of the two generator relations. Divide Υ into subsequences Υ_j as follows. Let Υ_1 be the longest initial subsequence of Υ which consists of regions all labeled by the same relator. Let Υ_j be the longest initial subsequence of $\Upsilon - (\Upsilon_1, \Upsilon_2, \dots, \Upsilon_{j-1})$ which consists of regions all labeled by the same relator.

To prove the proposition we use induction on the number of subsequences of Υ defined in the above fashion. If Υ has only one subsequence then this is exactly Lemma 15. Suppose the proposition is true for any strip of $n - 1$ subsequences. Let Υ be a strip of n subsequences. Let ω be the label on $ext(\Upsilon)$ and let v be the label on $base(\Upsilon)$. The edge e which is shared by $\Upsilon - \Upsilon_n$ and Υ_n is labeled by a single letter since the two regions which contain e in their boundary are both labeled by different relators.

If we delete Υ_n from the diagram M, then $\Upsilon - \Upsilon_n$ is a strip. Let M' be the diagram with Υ_n deleted and let Υ' be the corresponding strip. Let ω' be the label on $ext(\Upsilon', M')$ and let v' be the label on $base(\Upsilon', M')$. Then by induction we have that

$$|\omega'| \geq |v'| + 2 \quad \text{and} \quad \|\omega'\| \geq \|v'\| + 2.$$

Similarly, we can delete $\Upsilon - \Upsilon_n$ for the diagram to form a diagram M'' with a strip Υ'' corresponding to the subsequence Υ_n. Let ω'' be the label on $ext(\Upsilon'', M'')$ and let v'' be the label on $base(\Upsilon'', M'')$. Again we have that

$$|\omega''| \geq |v''| + 2 \quad \text{and} \quad \|\omega''\| \geq \|v''\| + 2.$$

Now we use the fact that the edge e is labeled by a single letter to get

$$|\omega| = |\omega'| + |\omega''| - 2 \geq (|v'| + 2) + (|v''| + 2) - 2 = |v| + 2 \quad \text{and}$$

$$\|\omega\| \geq \|\omega'\| + \|\omega''\| - 2 \geq (\|v'\| + 2) + (\|v''\| + 2) - 2 \geq \|v\| + 2.$$

This proves the proposition. $\square$

Each of the subsequences, defined in the proof above, share exactly one edge with the preceding and following subsequence. These shared edges must be labeled by a single letter. Using this fact it is easy to prove the following corollary.

Corollary 18. *Suppose that $H = \langle \mathscr{X} | \mathscr{R} \rangle$ is an Artin group of large type, and M is a $C(4)$–$T(4)$ reduced $\mathscr{R}$-diagram. Suppose that Υ is a strip of M. Let Υ_i, for $1 \le i \le n$, be the subsequences of the strip as defined in the proof of Proposition 17. Then for $2 \le i \le n-1$, $ext(\Upsilon_i)$ is labeled by at least two syllables and, $ext(\Upsilon_1)$ and $ext(\Upsilon_n)$ are each labeled by at least three syllables.*

The last lemma in this section gives an important link between syllable length and word length.

Lemma 19 (Appel and Schupp [2]). *Suppose $\omega \in \mathscr{A}_{ij}^{*}$ is equal to the identity in H_{ij} and $\omega = \omega_1\omega_2$.*

(1) $\|\omega\| \ge 2m_{ij}$.

(2) *If* $\|\omega_1\| \le m_{ij}$ *then* $|\omega_1| \le |\omega_2|$.

(3) *If* $\|\omega_1\| \le m_{ij}$ *then* $|\omega_1| < |\omega_2|$.

5. Thin equality diagrams and fellow travelers

An *equality diagram* for two words, ω and v, equal in the group $G = \langle \mathscr{X} | \mathscr{R} \rangle$, is an $\mathscr{R}$-diagram labeled by the word ωv^{-1}. We can think of $\omega(t)$ and $v(t)$ as paths along the boundary of the equality diagram. The 1-complex of an $\mathscr{R}$-diagram will not necessarily embed into the Cayley graph of the group. But it is clear that if there is a path between the points $\omega(t')$ and $v(t')$ of length k in the $\mathscr{R}$-diagram, then $d(\omega(t'), v(t')) \le k$ in the Cayley graph. So to determine whether paths are k-fellow travelers in the Cayley graph, we can concentrate on whether there is an equality $\mathscr{R}$-diagrams in which the paths are k-fellow travellers.

Let M be an equality diagram for the words ω and v. Let p_0 and p_f be the points of ∂M where the words begin and end, respectively. The diagram M is said to be *basic thin* if M is either a single region or M satisfies the following two properties:

(1) There are two distinguished regions, D_0 and D_f. These are the only simple boundary regions of M. For each of these regions the interior degree is one. Furthermore $p_0 \in openext(D_0)$ and $p_f \in openext(D_f)$.

(2) The remaining regions of M are all non-simple boundary regions. For any one of these regions, D, the interior degree is two, and $\partial D \cap \omega$ and $\partial D \cap v$ are each non-empty, consecutive portions of ∂M. Notice however that $\partial D \cap \omega$ or $\partial D \cap v$ may be only a single vertex.

The diagram will look like the one in Fig. 2. A *bridge* in an equality diagram is a portion of the boundary which does not lie on the boundary of any region and is

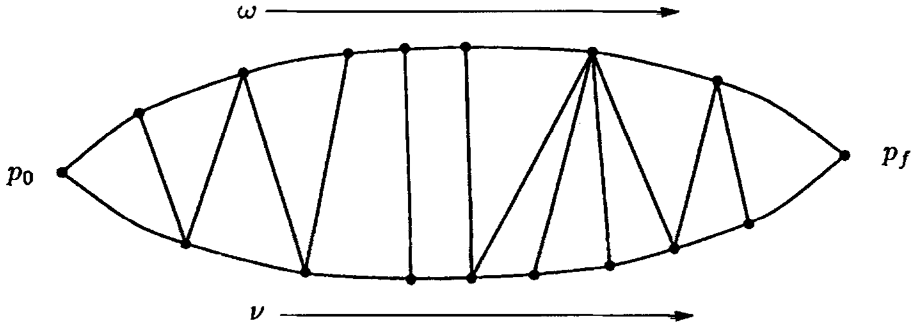

Fig. 2. Basic thin equality diagram.

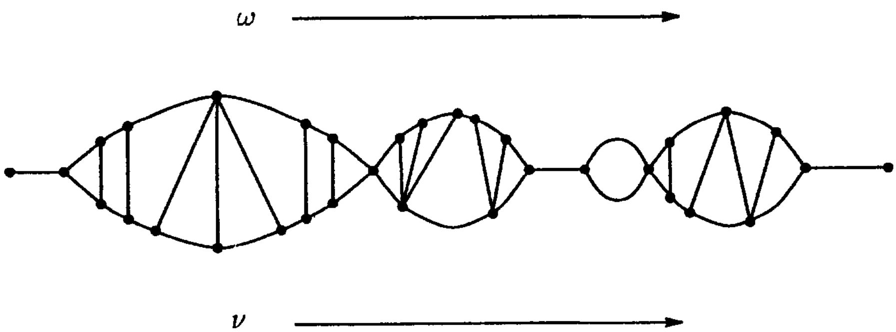

Fig. 3. Thin equality diagram.

labeled by a portion of both words. A bridge appears in the diagram as a long edge. For any bridge, let p_0 be the beginning point and p_f be the end point.

A *thin equality diagram* (Fig. 3) is a finite connected string of bridges and basic thin equality diagrams. These are strung together by connecting the point p_f of one to the point p_0 of the next.

Suppose two words, ω and v, in $\mathscr{A}^*$ are equal in the group G and have a thin equality $\mathscr{R}$-diagram, M. Furthermore, assume that $\mathscr{R}$ is finite and the maximum length of the relators is $2k$. Let p be a point of $\partial M \cap \omega$. If the point p is on a bridge, then p is a point of $\partial M \cap v$. If the point p is on the boundary of a region, D, then p is within k of $\partial D \cap v$. Thus, we have the following lemma.

Lemma 20. *Suppose $G = \langle \mathscr{X} | \mathscr{R} \rangle$ is a finitely related group with $2k$ the maximum length of a relator. Let ω and v be words in $\mathscr{A}^*$ which are equal in the group. If there exists a thin equality diagram for the words ω and v then the two paths ω and v in the Cayley graph $\Gamma_{\mathscr{A}}(G)$ lie in a k-Hausdorff neighborhood of each other. Furthermore, if the two words ω and v are geodesic paths in the Cayley graph then by Proposition* 1, *they are 2k-fellow travelers.*

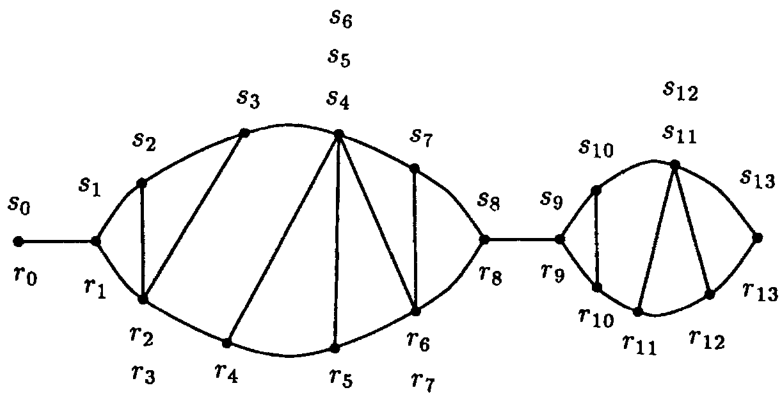

Fig. 4. An example of the labeling scheme.

A more general connection between thin equality diagrams and fellow travelers is shown in the next lemma.

Lemma 21. *Suppose $G = \langle \mathscr{X} | \mathscr{R} \rangle$ is a finitely related group with $2k$ the maximum length of a relator. Let ω and v be words in $\mathscr{A}^*$ which are equal in the group. Suppose that there exist a thin equality $\mathscr{R}$-diagram for ω and v such that all edges of M, which are on the boundary of a region, have maximum length k. Then at least one of the following is true:*

(1) *ω and v are $2k$-fellow travelers,*
(2) *there exist a path γ such that ω and γ are $2k$-fellow travelers, $\bar{\gamma} = \bar{\omega}$ and $|\gamma| < |\omega|$,*
(3) *there exist a path γ such that v and γ are $2k$-fellow travelers, $\bar{\gamma} = \bar{v}$ and $|\gamma| < |v|$.*

Proof (sketch). Let M be a thin equality $\mathscr{R}$-diagram for the words ω and v. The diagram M looks like the diagram in Fig. 3 above.

Label the vertices along the path ω with the sequence $\{s_0, s_1, \ldots, s_n\}$. The vertices are labeled in the order in which they appear along ω. The beginning point p_0 is labeled s_0. A vertex v is given multiple labels if $d(v) \geq 3$, in the following manner. If the preceding vertex has its largest label s_h and $d(v) = 2 + j$ then label the vertex v with $s_{h+1}, s_{h+2}, \ldots, s_{h+j}$. Label the vertices along v in the same manner with the sequence $\{r_0, r_1, \ldots, r_n\}$. An example of this labeling scheme is given in Fig. 4.

Let $l(\omega, i)$ be the length of the word ω up to the vertex s_i, and similarly let $l(v, i)$ be the length of the word v up to the vertex r_i. If $|l(\omega, i) - l(v, i)| \leq k$ for all i, then it is easy to show that the two paths ω and v are $2k$-fellow travelers. Suppose, on the other hand, that for some value of i, $|l(\omega, i) - l(v, i)| > k$. Let h be the first index for which this happens. Assume that $l(\omega, h)$ is larger than $l(v, h)$. Let γ be the label on the path that starts at p_0 and runs along v up to the vertex r_h, then passes over the edge between r_h and s_h and then continues along the remaining portion of ω to p_f. The path γ satisfies conclusion (2) of the lemma. If on the other hand, $l(v, h)$ is larger than $l(\omega, h)$, then in a similar manner we can find a path which satisfies conclusion (3) of the lemma. □

6. $C(4)$–$T(4)$ equality diagram for stripless words

Let G be a finitely generated group with presentation $\langle \mathscr{X} | \mathscr{R} \rangle$. In this section we examine $\mathscr{R}$-diagrams with the following properties:

(1) M is an equality diagram for the two words, α and β;

(2) M satisfies $C(4)$–$T(4)$; and

(3) no subword of α or β labels the entire exterior of a spike or strip.

Throughout this section we will assume that M satisfies these properties.

Let p_0 and p_f be the point of ∂M where the two words begin and end, respectively. If the diagram begins or ends with a bridge, remove the bridge or bridges and consider the remaining diagram. The remaining diagram will still satisfy the properties stated above. We will frequently abuse the notation and refer to edge paths in $\mathscr{R}$-diagrams by their labeling word. It will be clear from context when α refers to a boundary portion and when α refers to a word.

Lemma 22. *The points p_0 and p_f are not at vertices of M.*

Proof. If M is a single region then M has no vertices and the result is trivial. Suppose that M contains more than one region, and that p_0 is at a vertex of M. If M contains two or more singleton strips then at least one of α and β will label the exterior of one of the singleton strips, contradicting the hypothesis. If M contains only one singleton strip, D, then $p_f \in openext(D)$. Otherwise, α or β will label the exterior of the singleton strip. By Corollary 11, there are two other strips of M. Since p_0 is at a vertex of M, one of these strips will have its entire exterior label in one of the words α or β. This is again a contradiction.

The last possible case is that M has no singleton strips. By Corollary 11, M contains four strips. But again, because p_0 is a vertex of M one of the strips will lie so as to have its entire exterior label within one of the two words α or β. This is a contradiction. Thus P_0 is not a vertex of M.

A similar argument shows that p_f must not be a vertex of M. □

There are two distinguished regions of the diagram M. These regions include p_0 or p_f in their exterior and are denoted by D_0 and D_f, respectively.

Lemma 23. *For all boundary regions D of M, both $ext(D) \cap \alpha$ and $ext(D) \cap \beta$ are consecutive portions of ∂M.*

Proof. Suppose that, for some region D of M the boundary portion $\mathrm{ext}(D) \cap \alpha$ is not a consecutive portion of ∂M. Then there is a subdiagram K of M such that

$base(K) \subset \partial D$ and $ext(K) \subset \alpha$. Consider the subdiagram $K \cup D$ of M. By Corollary 11, $K \cup D$ has at least two disjoint strips. The region D can only be in one of these strips. Thus, the other strip has its entire boundary in α. But this contradicts the assumption that α does not label the exterior of any strips. □

Lemma 23 shows that any non-simple boundary region of M touches both words. The non-simple boundary regions therefore make up the thin part of the diagram. We can now begin to get a picture of the diagram M. It will contain a number of thick subdiagrams (these contain only simple boundary regions) strung together by non-simple boundary regions and bridges. We need to examine the thick subdiagrams of M. The first thick subdiagram is denoted by M_{th}.

The subdiagram M_{th} can be found in the following manner. Start with the diagram M. If the region D_0 is a singleton strip remove it from the diagram. Call the resulting diagram M_1. There are two possible situations depending on whether $base(D_0, M)$ is an edge or a vertex. If $base(D_0, M)$ is an edge, then there is a distinguished region, D_1, in M_1 which includes in its exterior the old $base(D_0, M)$. On the other hand, if $base(D_0, M)$ is a single vertex, then label the associated point of M_1, with p_0. If p_0 lies at the end of a bridge remove the bridge from M_1. If p_0 lies on a region then just as in Lemma 22 we can show that it does not lie at a vertex of M_1. Therefore, again there is a distinguished region, D_1, with p_0 in $openext(D_1, M_1)$. In both of the cases we end up with a new diagram M_1 with a new distinguished region D_1. Now continue this process. If the region D_1 is a singleton strip of M_1 remove it to get a new diagram M_2 with a distinguished region D_2. If the original diagram M was thin then this process will end with a diagram M_n which is a single region. If M was not thin then we must eventually find a diagram M_n in which the region D_n is not a singleton strip. Denote this region F. Now follow the boundary words α and β further on the diagram M_n until the first boundary region after F which touches both words. Call this region L. Finally, remove from M_n all the regions after the region L. This is now the diagram M_{th}, the first thick section of M.

Now the goal is to examine M_{th}. We pay special attention to the boundary regions and how they fit together. To begin with, note a few obvious properties which follow from the construction of M_{th}.

Lemma 24. *Let α' and β' be the subwords of α and β, respectively, which label portions of ∂M_{th}. Let ε be the label on the portion of ∂M_{th} which was the base of the last removed singleton strip. Let δ be the label on the $ext(L, M_{\text{th}})$.*

(1) *The word ε may label as little as a vertex or as much as a single edge.*
(2) *The only regions in M_{th} which touch both α' and β' are F and L.*
(3) *All the boundary regions of M_{th} are simple.*
(4) *There are no singleton strips or spikes on M_{th}.*
(5) *There are no cut vertices in M_{th}.*
(6) *The diagram M_{th} has more than one region.*

Recall that M is $C(4)$–$T(4)$; therefore M_{th} is also $C(4)$–$T(4)$. The $C(4)$–$T(4)$ hypothesis implies that the degree of interior regions, and the degree of interior vertices must be at least four. The next lemma states that the added hypothesis that α and β do not label strips forces these degrees to be exactly four.

Lemma 25. *The diagram M_{th} satisfies the following properties:*
(1) $\sum^{\bullet}(3 - i(D)) = 4$.
(2) *All interior regions have degree four.*
(3) *All interior vertices have degree four.*
(4) *All boundary regions have interior degree less than or equal to four.*

Proof. This follows from Lyndon's curvature theorem. Notice that M_{th} satisfies all the hypothesis of Corollary 10. Therefore, the diagram M_{th} satisfies the following equality:

$$4 = \sum^{\bullet}(3 - i(D)) + \sum^{\circ}(4 - d(D)) + \sum^{\circ}(4 - d(v)).$$

Because M_{th} is $C(4)$–$T(4)$, all interior vertices and all interior regions have degree at least four. Thus, $\sum^{\circ}(4 - d(D)) \leq 0$ and $\sum^{\circ}(4 - d(v)) \leq 0$. If either of these is negative then $\sum^{\bullet}(3 - i(D)) > 4$. But for this sum to be greater than four, M_{th} must contain at least five strips. There is no way for M_{th} to contain five strips without the entire exterior of at least one strip being contained in one of α' of β'. This is a contradiction. Therefore $\sum^{\circ}(4 - d(D)) = 0$ and $\sum^{\circ}(4 - d(v)) = 0$. This proves parts (1)–(3).

Now suppose that there is a boundary region D_1 with $i(D_1) \geq 5$. Since $\sum^{*}(3 - i(D)) \geq 4$ we have

$$\sum_{D \neq D_1}^{\bullet}(3 - i(D)) \geq 6.$$

This again forces five strips and thus a strip with the entire exterior within α' or β'. So again this is a contradiction. This proves part (4). □

Lemma 26. *The regions F and L of M_{th} satisfy, $i(F) = i(L) = 2$, F and L both begin compound strips in the clockwise and the counterclockwise directions, and there are no other strips in M_{th}.*

Proof. By Corollary 11, M_{th} contains four or more strips. But M_{th} cannot contain five strips without containing a strip whose entire exterior is within α' or β'. Thus, M_{th} has exactly four compound strips. The only way that four strips can fit into M_{th} without one of the words α' and β' labeling the exterior of one of them is if F and L both begin compound strips in both directions. □

Lemma 27. *One of $ext(F, M) \cap \alpha'$ or $ext(F, M) \cap \beta'$ is more than a single vertex.*

Proof. In the diagram M_{th}, $i(F) = 2$. Recall if any region which touched F was removed in the construction of M_{th} the removed region was a singleton strip. Thus, that region could have only covered a piece of $ext(F, M_{\text{th}})$. Since the original diagram M satisfied $C(4)$, and $i(F) = 2$, $ext(F, M_{\text{th}})$ is more than one piece. □

To get a better picture of the diagram M_{th} we will examine more closely the boundary regions. Define $\nabla = \{B_1, B_2, \dots, B_{n_\nabla}\}$ to be the sequence of boundary regions of M_{th}, encountered consecutively in the counterclockwise direction, starting just after F and ending just before L. Similarly, define $\Delta = \{A_1, A_2, \dots, A_{n_\Delta}\}$ to be the sequence of boundary regions of M_{th}, encountered consecutively in the clockwise direction, starting just after F and ending just before L. Notice that the set of all boundary regions of M_{th} is equal to the disjoint union of F, L, ∇ and Δ.

In the sense of Lyndon's curvature theorem, each boundary region adds curvature to the diagram. Using the convention that the total curvature of the diagram is four, define the *curvature* of a boundary region to be

$$\tau(D) = (3 - i(D)).$$

In this same sense a sequence of consecutively encountered boundary regions (reading in the clockwise or counterclockwise direction) adds curvature to the diagram. If Υ is a sequence of boundary regions let $\tau(\Upsilon)$ denote $\sum_{D \in \Upsilon}(3 - i(D))$.

Lemma 28. *Both* ∇ *and* Δ *have curvature one, i.e.* $\tau(\nabla) = 1$ *and* $\tau(\Delta) = 1$. *Furthermore, if* Υ *is an initial subsequence of* ∇ *or* Δ, *then* $\tau(\Upsilon) = 0$ *or* 1.

Proof. Let Υ be an initial subsequence of ∇ or Δ. Suppose that $\tau(\Upsilon) > 1$. Then Υ must contain a region D_1 with $\tau(D_1) = 1$ followed by a sequence (possibly empty) of regions with curvature zero and then a region D_2 with $\tau(D_2) = 1$. But the regions D_1 through D_2 form a strip whose exterior is entirely within either α' or β'. This is a contradiction. Thus $\tau(\Upsilon) \leq 1$.

By Lemma 25, $\sum^\bullet(3 - i(D)) = 4$. Thus, we have

$$\tau(F) + \tau(L) + \tau(\nabla) + \tau(\Delta) = 4.$$

By Lemma 26, $\tau(F) = \tau(L) = 1$. Thus, $\tau(\nabla) + \tau(\Delta) = 2$. Therefore, since neither quantity is greater than one, $\tau(\nabla) = \tau(\Delta) = 1$.

Now suppose that $\tau(\Upsilon) < 0$. Let Υ' be the sequence of boundary regions obtained by truncating Υ off the front of ∇ or Δ, whichever is appropriate. Since $\tau(\Upsilon) < 0$, we must have $\tau(\Upsilon') > 2$. But as in the first paragraph of this proof Υ' must contain a strip, which is a contradiction. Therefore, $\tau(\Upsilon) \geq 0$. □

The last two lemmas state that if side regions are ignored, ∇ and Δ begin and end with corners. Furthermore, within each of these sequences corners and inner corners are encountered alternately. Fig. 5 is a sketch of a possible diagram M_{th}.

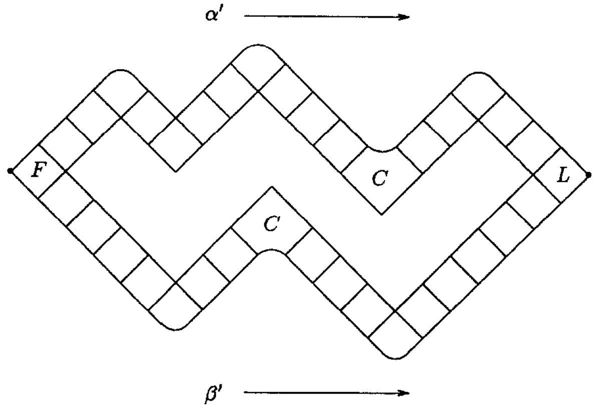

Fig. 5. The boundary regions of M_{th}.

7. Refuting the strip words

In Sections 3–6 we examined $\mathscr{R}$-diagrams. In particular, Section 5 contains the definition of thin equality diagrams and Section 6 contains an analysis of the structure of non-thin $C(4)$–$T(4)$ equality diagrams. We are now ready to prove that if H is an Artin group of extra-large type then the language $L(H)$ is regular. The method relies on a lemma of Davis and Shapiro. We supply a proof, due to Davis and Shapiro. Let $\overline{L(G)}$ denote the complement to the language $L(G)$ in $\mathscr{A}^*$.

Lemma 29. *Let G be a group given by a presentation $\langle \mathscr{X} \mid \mathscr{R} \rangle$, with the generating set $\mathscr{X}$ finite. Let $\mathscr{A}$ be the set of semigroup generators and let $\prec$ be a regular ordering of the elements of $\mathscr{A}^*$. The language*

$$L(G) = \{v \in A^* \mid v \text{ is } \prec \text{ minimal for } \bar{v}\}$$

is regular if there is a constant number k, such that for every $\omega \in \overline{L(G)}$ there is a k-fellow traveler v, with $v \prec \omega \in \mathscr{A}^*$ and $\bar{v} = \bar{\omega} \in G$.

Proof. To begin with, we will show that the language $L_N = \{(v, \omega) \mid \bar{v} = \bar{\omega}$ and v and ω are k-fellow travelers$\}$ is regular. We do this by building a finite state automaton, M_N, that accepts this language. Let $B(N)$ be the set of all elements of G that are a distance of N, or less, from the identity in the Cayley graph of G. The states of M_N consist of a fail state together with one state for each element of $B(N)$. The start state is 1. Suppose the automaton is in state $g \in B(N)$ and is reading the pair of letters $(a_1, a_2) \in \mathscr{A} \times \mathscr{A}$. (The padding element should be taken to be 1.) If $\overline{a_1^{-1}\, g \bar{a}_2}$ is in $B(N)$, then the transition function changes the machine to state $\overline{a_1^{-1}\, g \bar{a}_2}$. If however, $\overline{a_1^{-1}\, g \bar{a}_2}$ is not in $B(N)$, then the transition function changes the machine to the fail state. Once the machine is in the fail state it remains there. The only accept state is 1. It is easy to

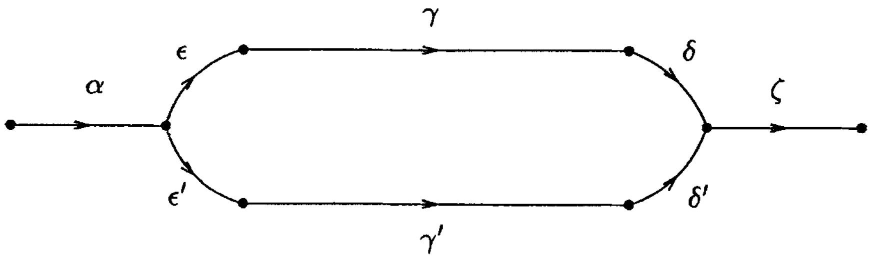

Fig. 6. k-parallel paths.

check that the M_N accepts a pair (v, ω) if and only if $(v, \omega) \in L_N$. Therefore, L_N is regular.

By hypothesis $L_{\prec} = \{(v, \omega) \mid v \prec \omega\}$ in regular. Let p_2 denote the projection on the second factor. It is a standard result that the projection of a regular language is regular, see either [6] or [10]. Finally, we have

$$L(G) = p_2(L_N \cap L_{\prec}).$$

Therefore, the language $L(G)$ is regular. □

We say that the word v *k-refutes* the word ω, relative to the ordering $\prec$, if v and ω satisfy the following properties: $\bar{v} = \bar{\omega}$ in G, $v \prec \omega$, and v and ω are k-fellow travelers.

Suppose that H is an Artin group of extra-large type for which the largest finite m_{ij} value is m. In Sections 7–10, we show that all words $\omega \in \overline{L(H)}$ are $2m$-refuted. We begin this section by focusing on some special cases where the word ω contains some simple types of subwords. In Section 8, this is extended to the case where ω contains two generator subwords that are not geodesics in their respective two generator group H_{ij}. In Section 9, this is again extended to words which contain any non-geodesic subword. Finally, in Section 10 we prove the main result, Theorem A.

One thing that must be done in order to show that a word $2m$-refutes another is to prove that the words are $2m$-fellow travelers. In the rest of this paper we will frequently show that two words have some very specific properties and will then claim that they are $2m$-fellow travelers. To this end we define k-parallel words to be words with these specific properties.

Suppose that ω and v are two words in $\mathscr{A}^*$, with subwords, α, ε, ε', δ, δ', γ, γ', and ζ, such that

(1) $\bar{\omega} = \bar{v}$.
(2) $\omega = \alpha\varepsilon\gamma\delta\zeta$ and $v = \alpha\varepsilon'\gamma'\delta'\zeta$.
(3) $|\varepsilon|$, $|\varepsilon'|$, $|\delta|$, $|\delta'|$ are all less then or equal to k.
(4) γ and γ' have the same length and are k-fellow travelers.

Then we say that the words (or paths) are *k-parallel.*

Two paths that are k-parallel are in fact $2k$-fellow travelers. This is easy to show using the fact that $||\varepsilon| - |\varepsilon'||$ and $||\delta| - |\delta'||$ are both less than or equal to k. We state this fact as a lemma for future reference.

Lemma 30. *If two words ω and v are k-parallel then they are $2k$-fellow travelers.*

Recall that a *strip word* is a word which can label the exterior of a strip in a $C(4)$–$T(4)$ $\mathscr{R}$-diagram. We prove that for any two generator Artin group or Artin group of large type, all words in $\mathscr{A}^*$ which have strip subwords are $2m$-refuted, where $m = \max\{m_{ij} | m_{ij} \neq \infty\}$. Start by eliminating some special cases. In Lemma 31, a Dehn word refers to a subword of a relator that has length more than half the length of the relator.

Lemma 31. *Suppose that H is any Artin group. Let $m = \max\{m_{ij} | m_{ij} \neq \infty\}$. If $\omega \in \mathscr{A}^*$ contains a subword which is either an inverse pair, a Dehn word or a far-corner then there exist a word v in $\mathscr{A}^*$ which $2m$-refutes ω.*

Proof (sketch). We will do the case when ω contains a far-corner subword. The other cases are similar. Replacing the half relator in the far-corner by the corresponding half relator will produce a word v of equal length that represents the same group element. There is a one region equality diagram for these two words. It starts with a bridge up to the point of the far-corner. At this point there is one region with the high and the low words passing around either side. Then the paths meet again and continue along a bridge until the end of the diagram. Because the half relators have length less than or equal to m, these paths are clearly m-fellow travelers. The only thing left to show is that $v \prec \omega$, but this follows directly from Corollary 6. □

Proposition 32. *Suppose that H is either a two generator Artin group or an Artin group of large type. Let $m = \max\{m_{ij} | m_{ij} \neq \infty\}$. If $\omega \in \mathscr{A}^*$ contains a strip word then there exist a word $v \in \mathscr{A}^*$ which $2m$-refutes ω.*

Proof. Let ω be a word which contains a strip word. By Lemma 31 we can assume that ω does not contain any inverse pairs, Dehn words or far-corners. Notice that a singleton strip word is a Dehn word.

Case 1: Assume that ω contains a strip word $\gamma \in \mathscr{A}_{ij}^*$, that is associated with a strip made up of regions all labeled by the same relator. Let $\gamma' \in \mathscr{A}_{ij}^*$ be the word that would label the base of the strip. Let v be the word obtained by replacing the occurrence of γ in ω with the word γ'. Clearly, $\bar{\omega} = \bar{v}$ and by Lemma 15, $|v| < |\omega|$ so that $v \prec \omega$. Thus, we need only to show that ω and v are $2m$-fellow travelers.

Consider the equality diagram for v and ω in Fig. 7. The regions for this diagram come from the regions of the strip associated with γ.

By finding the positions of the poles for each region we will be able to show that Lemma 30 applies. First examine the last region, D_n. Since we have assumed that the

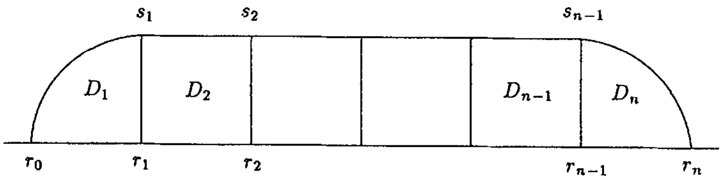

Fig. 7. The equality diagram for ω and v.

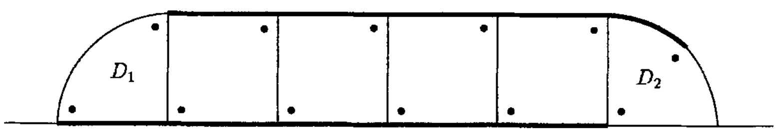

Fig. 8. The poles in the equality diagram for v and ω.

strip is not a singleton strip, we know that there is a region D_{n-1} before D_n. The word ω labels a portion of $ext(D_{n-1})$ and the vertex s_{n-1} has degree three. Therefore, D_n cannot have both poles in $ext(D_n) \cap \omega$, this would force ω to contain a Dehn or far-corner subword. The two other edges of D_n are labeled by pieces. Pieces never contain poles. So one of the poles must be at the vertex r_{n-1}. The other must be in $ext(D_n) \cap \omega$ but not at either boundary vertex.

The region D_{n-1} must have a pole at vertex s_{n-1}, by Lemma 12. If D_{n-1} is not the first region of the strip then the other pole of D_{n-1} cannot also lie on $ext(D_{n-1}) \cap \omega$ because this would imply that ω contains a Dehn or far-corner subword. Therefore, since all the other edges of D_{n-1} are labeled by pieces, the other pole for D_{n-1} is at the vertex r_{n-2}. Continuing this same line of argument we can show that the poles for each region are as pictured in Fig. 8.

Because of the positioning of the poles, for $2 \leq j \leq n$, the label on $\partial D_j \cap \omega$ has exactly the same length as the label on $\partial D_{j-1} \cap v$. In fact, if m_{ij} is even then the labels, read moving to the right, are identical. If m_{ij} is odd, then the labels are related by the correspondence $x_i \leftrightarrow x_j$, $x_i^{-1} \leftrightarrow x_j^{-1}$. In Fig. 8 the dark edges represent the portions of the two paths that match in the sense just described.

By examining Fig. 8 we can see that the two paths v and ω are m-parallel. The dark edges in the diagram in Fig. 8 correspond to the subwords γ and γ' of the definition of k-parallel. Therefore, the two paths ω and v, in the Cayley graph, are $2m$-fellow travelers. Thus, v $2m$-refutes ω.

Case 2: Now we consider the case that the strip word is not a two generator strip word. Recall from the proof of Proposition 17 that the strip can be divided into a set of subsequences. Let Υ be the last of these subsequences, with regions labeled by the relator r_{ij}. Let λ be the label on $\partial\Upsilon \cap \omega$ and let λ' be the label on the remaining portion of $\partial\Upsilon$. Let v be the word obtained by replacing the occurrence of λ in the word ω by λ'. Clearly $\bar{\omega} = \bar{v}$.

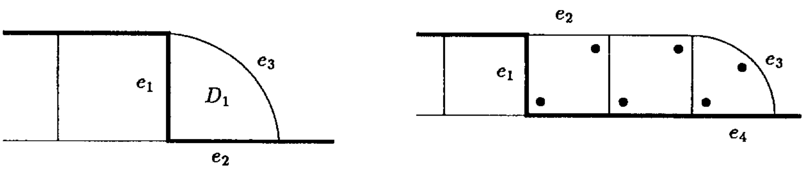

Fig. 9. The subsequence Υ.

We will first examine the case when Υ is only one region, pictured in Fig. 9. The edge e_1 must be labeled by a single letter. This is because this edge is shared by two regions which are labeled by different relators. The edge e_2 is labeled by a piece, so its label must have length less than m_{ij}. Therefore, the edge e_3 is either labeled by a Dehn word or a half relator. We have assumed that ω contains no Dehn words, so e_3 is labeled by a half relator. But, by Corollary 18 both syllables of the preceding subsequence of the strip will appear on the exterior of the strip. Thus, this is the configuration of a type II far-corner. So by Lemma 31, Υ cannot be a single region.

Fig. 9 also depicts Υ with more than one region. The poles must lie as they are placed in the figure. This can be shown by a similar argument as that used in Case 1. Now again, by Lemma 30, ω and v are $2m$-fellow travelers. All that is left to do is show that $v \prec \omega$.

The length of the edge e_1 must be 1 since it is labeled by a single generator (Note e_1 is shared by regions on different relators.) Therefore, the length of e_2 is $m_{ij} - 1$. Let e_3 represent the half edge from the pole to the vertex shared with edge e_4.

Suppose that $|e_3| > 1$, then $|e_4| < m_{ij} - 1$. Thus, we have that

$$|v| = l + |e_1| + |e_4| \leq l + m_{ij} = (m_{ij} - 1) + l + 1 < |e_2| + l + |e_3| = |\omega|.$$

Since $|v| < |\omega|$ it is clear that $v \prec \omega$.

Suppose however that $|e_3| = 1$ and $|e_4| = m_{ij} - 1$, then $|\omega| = |v|$. In this case we need to show that $\mathscr{B}(v) < \mathscr{B}(\omega)$. Let Υ' be the subsequence of the strip that immediately precedes Υ.

By Corollary 18 both generators appear in the label along $ext(\Upsilon')$. Since both generators of the relator labeling Υ' appear in ω we can see that ω contains an excessive word. By Corollary 6, $\mathscr{B}(v) \prec \mathscr{B}(\omega)$. This completes the proof. $\square$

8. Refuting two generator non-geodesics

Consider the Artin group H with language $L(H) \subset \mathscr{A}^*$. For any two distinct generators x_i and x_j there is an Artin group H_{ij} with language $L(H_{ij}) \subset \mathscr{A}^*_{ij}$. A word $\omega \in \mathscr{A}^*$ is called a *two generator geodesic* if it is an element of one of the languages $\mathscr{A}^*_{ij}$, and is a geodesic in the Cayley graph of H_{ij}. The set of words of particular interest in this section are the two generator non-geodesic words. This is the set of all two generator words, on any two generators from $\mathscr{X}$, that are not two generator geodesics.

Proposition 33. *Suppose that H is an Artin group of large type, or a two generator Artin group, with $m = \max\{m_{ij} \mid m_{ij} \neq \infty\}$. Then any word in $\mathscr{A}^*$ containing a two generator non-geodesic subword is 2m-refuted.*

Suppose $\omega \in \mathscr{A}^*$ contains a two generator non-geodesic subword. To prove Proposition 33 we can assume that ω does not contain any inverse pairs, Dehn words, far-corners, or strip words. Proposition 32 states that any word containing one of these as a subwords is already known to be $2m$-refuted. Pick ω' to have minimal length among all two-generator non-geodesic subwords of ω. Suppose that ω' is in $\mathscr{A}^*_{ij}$. Note that the value m_{ij} for the group H_{ij} must be finite. If m_{ij} were infinite then H_{ij} would be the free group on two generators and all non-geodesics will contain an inverse pair, but we have assumed that ω, and thus that ω' does not contain inverse pairs. Notice also that all proper subwords of ω' are two generator geodesics. Let v' be a word in $L(H_{ij})$ such that $\bar{v}' = \bar{\omega}'$. The word v' is a geodesic word; therefore, $|v'| < |\omega'|$. Let v be the word obtained by replacing the occurrence of ω' in ω with the word v'.

Let M be an equality $\mathscr{R}_{ij}$-diagram for the word v' and ω'. If the diagram M is a thin equality diagram then Proposition 33 follows directly from Proposition 21. Therefore, we need only consider the case that when M is not thin. The diagram M is a $C(4)$–$T(4)$ because the two generator Artin group H_{ij} satisfies $C(4)$–$T(4)$. Notice also, that the words ω' and v' do not label the exterior of a spike or a strip in ∂M because we have assumed ω contains no inverse pairs or strip words. Therefore, M is a diagram like the diagrams discussed in Section 6. As in Section 6, M contains a first thick subdiagram M_{th}. The central objective now is to locate the poles for each of the boundary regions of M_{th}. With this information we will be able to locate a path through M_{th} whose label $2m$-refutes ω.

In order to carefully examine M_{th} we introduce the following terminology. Let ω'' and v'' be the labels on $\omega' \cap \partial M_{\text{th}}$ and $v' \cap \partial M_{\text{th}}$, respectively. Let F and L be the first and last regions of M_{th}. Let $\Delta = \{W_1, W_2, \dots, W_s\}$ and $\nabla = \{V_1, V_2, \dots, V_r\}$ be the sequences of boundary regions along the top and bottom of the subdiagram M_{th}. Typically, M_{th} will look like the diagram in Fig. 5.

The next three lemmas describe the positioning of the poles for the boundary regions of M_{th}. The first is a short technical lemma that will be cited several times in the rest of this paper.

Lemma 34. *Let $\Upsilon = \{D_k \mid k = 1, \dots, n\}$ be a consecutive subsequence from Δ or ∇. Suppose that: Υ is at least two regions long; the last region of Υ, D_n, is a corner; the rest of the regions are sides; and the ext(Υ) is entirely within ω'' or v''. Then the following are true.*

(1) *openext(D_n) contains a pole.*

(2) *The other pole of D_n is at the interior vertex.*

(3) *For each $1 \le k < n$, ext(D_k) contains exactly one pole which is located at the exterior vertex shared by the regions D_k and D_{k+1}.*

(4) *For each $2 \le k < n$ the base(D_k) contains exactly one pole which is located at the interior vertex shared by the regions D_{k-1} and D_k.*

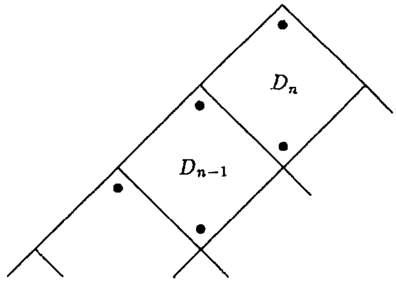

Fig. 10. Poles of a corner and preceding side regions.

Proof. Without loss of generality, assume that Υ is a subsequence of Δ and $ext(\Upsilon)$ is contained in ω''. First consider the corner region, D_n. Notice that the word ω'' begins before the first vertex of $ext(D_n)$. This vertex has degree three. Therefore, the last letter of ω'' (or v'') before the word labeling $ext(D_n)$ is the same as the first letter labeling $ext(D_n)$. If there is no pole at the interior vertex of D_n then both poles lie in $ext(D_n)$. But this would imply that the word ω'' contains a Dehn or far-corner subword contrary to our assumption. Therefore, there is a pole at the interior vertex of D_n. The other pole of D_n must be in $openext(D_n)$ because the interior edges of D_n are labeled by pieces.

The vertex shared by D_{n-1} and D_n must be a pole for the region D_{n-1} by Lemma 12. The proof is done if Υ is only two regions long. However, if Υ is longer, then we can determine the position of the poles for the rest of the side regions as follows. Suppose that there is another side region D_{n-2} before D_{n-1} in Υ. If the other pole of D_{n-1} is in $ext(D_{n-1})$ then ω'' (or v'') contains a Dehn or far-corner subword. Therefore, the second pole of D_{n-1} is at one of the two interior vertices. Since the interior edges are labeled by pieces the second pole must be at the interior vertex shared by D_{n-2} and D_{n-1}. By continuing in this fashion we can work through all the side regions and prove the lemma. $\square$

In Fig. 5, the two regions marked by C are inner corners with exterior edges. In Lemma 35 it will be shown that this is not possible, all inner corners have only a vertex on ∂M_{th}. Thus, inner corners have exactly four edges and four vertices. More importantly, Lemma 35 states where the poles for each boundary region lie. If a pole appears in the open exterior of a region we add a vertex at that point. Using this convention, vertices will be added to all corner regions (except F and L) and only to corner regions, so that all regions, except F and L, will have exactly four vertices and four edges. We assign the directions north, south, east, and west to the four vertices of each boundary region as would be done on a standard map. It is easy to verify that this is well defined.

Lemma 35. *Every region of Δ and ∇ has degree four and the poles for each region lie at the north and south vertices.*

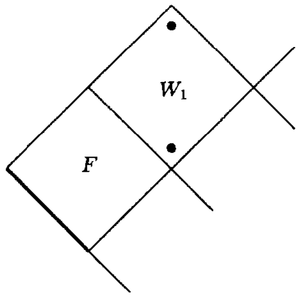

Fig. 11. W_1 is a corner.

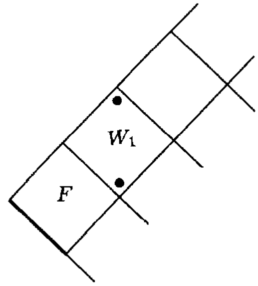

Fig. 12. W_1 is a side.

Proof. One of $ext(F) \cap \omega''$ or $ext(F) \cap v''$ must include an edge. We will assume that $ext(F)$ has an edge in ω''. The other case is treated in the same manner. Let $\Delta = \{W_i\}$ and $\nabla = \{V_i\}$.

Suppose that W_1 is a corner as in Fig. 11. Then $ext(W_1)$ contains at least one pole. If the boundary vertex shared by F and W_1 is a pole then both poles of W_1 are on $ext(W_1)$, since interior edges are labeled by pieces. Similarly, if the vertex shared by W_1 and W_2 is a pole for W_1 then again both poles are on $ext(W_1)$. But W_1 cannot have both poles on $ext(W_1)$, since this would imply ω'' contains a far-corner or Dehn word (recall that we are assuming that $ext(F)$ contains an edge in ω''). Therefore, one of the poles for the region W_1 is at the vertex in $openbase(W_1)$. The other pole must lie in $openext(W_1)$ because interior edges are labeled by pieces. By convention a vertex is added at this pole. Therefore, the region W_1 has exactly four edges and has north–south poles.

Suppose that W_1 is a side region as in Fig. 12. Recall that $ext(F)$ contains an edge in ω''. Therefore, just as in Lemma 34, it can be shown that the poles are north and south. Because W_1 cannot be a inner corner, this proves the proposition for the region W_1.

Now an induction proof will show that the remaining boundary regions also satisfy the proposition. First, consider the regions in Δ. Suppose the first n regions of Δ satisfy the lemma. The region W_{n+1} is either a corner, side, or inner corner region. Look at each case separately.

If W_{n+1} is a side then either, W_{n+1} follows a corner or it follows an inner corner. Consider the case that W_{n+1} follows a corner, as in Fig. 13. The boundary vertex shared by W_n and W_{n+1} is not a pole for W_n by induction. Therefore, this vertex is a pole for W_{n+1}, by Lemma 12. Notice that $ext(W_{n+1})$ cannot contain another pole or else ω'' would contain a far-corner or Dehn word. The other pole must therefore be at the south vertex. If the region W_{n+1} follows an inner corner as shown in Fig. 14, then the poles are north and south, by Lemma 34.

Now suppose that W_{n+1} is a corner. The boundary vertex shared by W_n and W_{n+1} is not a pole for W_{n+1} by Lemma 12. If $ext(W_{n+1})$ contains two poles, then ω contains a Dehn word. Thus, one of the poles is at the interior vertex shared by regions W_n and

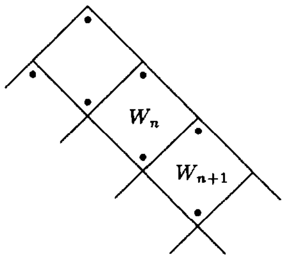

Fig. 13. W_{n+1} follows a corner.

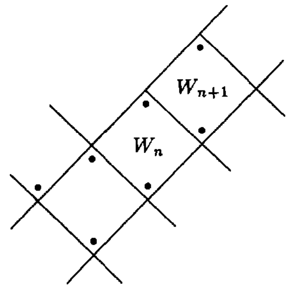

Fig. 14. W_{n+1} follows an inner corner.

W_{n+1}. Now the other pole must be within $openext(W_{n+1})$. Therefore, W_{n+1} satisfies the conclusion of the lemma.

Suppose W_{n+1} is an inner corner. Suppose $ext(W_{n+1})$ contains an edge. Then the vertex shared by W_n and W_{n+1} is a pole for W_{n+1}, by induction and Lemma 12. The edge in $ext(W_{n+1})$ cannot contain another pole or ω'' would contain a far-corner or Dehn subword. Thus, the vertex shared by W_{n+1} and W_{n+2} is not a pole for W_{n+1}. But now the same proof as in Lemma 34 shows that the next corner region of Δ would have both poles on its exterior. This forces a far-corner or Dehn subword in ω, which is a contradiction. Therefore, $ext(W_{n+1})$ does not contain an edge.

Thus, W_{n+1} has four edges which are all interior edges. Because of the positioning of the poles in the region W_n, Lemma 13 implies that the poles for W_{n+1} are north and south. Thus, all the regions in Δ satisfy the lemma.

In order to prove the lemma for the regions in ∇, first consider the region V_1. The vertex shared by the regions W_1, F and V_1 must be an interior vertex or else the diagram would contain a singleton strip. By Lemma 14, this vertex is a pole for V_1 since it is a pole for W_1. If V_1 is a corner, the fact that interior edges are labeled by pieces shows that there is a pole within $openext(V_1)$. So in this case V_1 satisfies the lemma. If V_1 is a side then Lemma 34 shows that the other pole is at the boundary vertex shared by V_1 and V_2. And again V_1 satisfies the lemma.

The rest of the regions in ∇ satisfy the lemma by a similar induction proof used for the regions of Δ. □

Lemma 36. *The poles of F and L are north and south.*

Proof. First consider the region F, pictured in Fig. 15. There may be a portion of $ext(F)$ which is not labeled by either of the words ω'' and v''. This portion is denoted by ε. If there is such a portion it will have come from the base of a singleton strip. Therefore, the label on ε is a piece. Either $ext(F)\cap\omega''$ or $ext(F)\cap v''$ contains an edge. We consider the case that $ext(F)\cap\omega''$ contains an edge; the proof in the other case is very similar. Because ω'' is freely reduced, Lemma 12 implies there is a pole at vertex v_0.

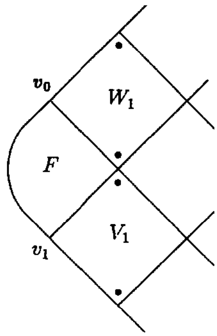

Fig. 15. The region F.

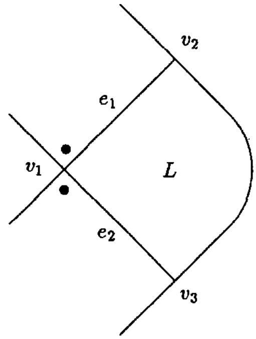

Fig. 16. The region L.

If $ext(F)\cap v''$ also contains an edge then again, by Lemma 12, the second pole must lie at vertex v_1. The only case left is when $ext(F)\cap v''$ does not contain an edge. If the other pole is in $ext(F)\cap\omega''$ then it must be at the end of ε. Otherwise the word ω'' would contain a Dehn word. If however the pole is at the end of ε, so that $ext(F)\cap\omega''$ is labeled by a half relator, then the label starting at ε along ω'' up to and including the exterior of the first corner of V, is a strip word. It may not appear as the boundary of a strip in this diagram, but notice that the only portion of ∂F not included in this word is one piece. However, we have assumed that ω'' does not contain strip words. Therefore, the other pole must be at vertex v_1, as claimed.

Now examine the region L, pictured in Fig. 16. We want to show that the two poles are at the vertices v_2 and v_3. The argument will depend on $i(L, M)$, the number of interior edges of L relative to the diagram M. First, note that the two edges e_1 and e_2 will be interior edges in M. Suppose that $i(L, M) = 2$ or 3. Then $ext(L)\cap\omega''$ includes an edge or $ext(L)\cap v''$ includes an edge. The argument in these two cases is very similar to the proof that was given above for the region F.

Suppose that $i(L, M) \geq 4$. In this case the region L must start a second thick subdiagram, M^2_{th}, of the diagram M. The second thick subdiagram must look very much like the first thick subdiagram. The lemmas in Section 6 apply to this subdiagram also. By Lemma 26, $i(L, M) = 4$ since in the second thick subdiagram $i(L, M^2_{\text{th}}) = 2$. If $ext(L)\cap\omega'$ includes an edge, say λ then there must be poles at both ends of λ by Lemmas 12 and 34. But this means that ω' contains a far-corner, a contradiction. Therefore, $ext(L)\cap\omega'$ is only a vertex. By similar methods $ext(L)\cap v'$ is only a vertex. Now it is clear that the poles must be at v_2 and v_3 since L cannot have a pole at v_1 by Lemma 14. $\square$

Fig. 17 is a sketch of a possible diagram M_{th}. Notice that the positioning of the poles shows that the labels on the two heavy lined paths are of equal length as are the paths on the dotted lines. With this information we can now prove Proposition 33.

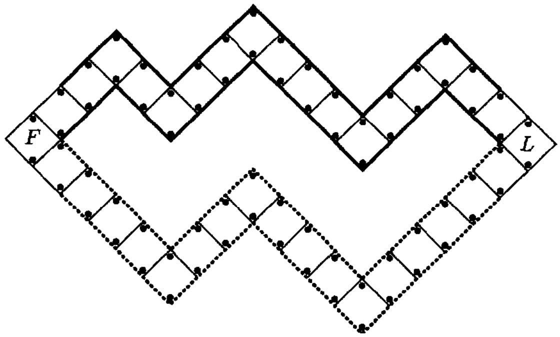

Fig. 17. The subdiagram M_{th} with poles shown.

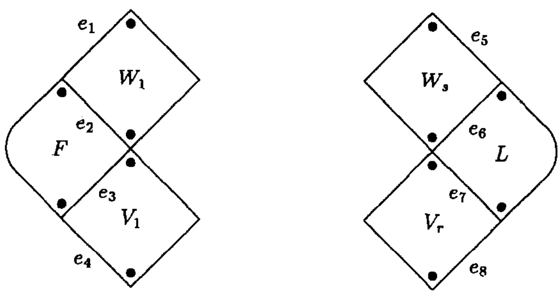

Fig. 18. The regions F and L.

Proof of Proposition 33. If the diagram M is thin then by Lemma 21, there is a word that $2m$-refutes ω. Therefore, assume that M is not thin. There is a first thick subdiagram, M_{th}. Let γ be the label on $ext(\Delta)$, and let γ' be the label on $base(\Delta)$. Let λ be the word obtained by replacing the occurrence of γ in ω by γ'. We will now show that λ $2m$-refutes ω.

By the positioning of the poles (see Fig. 17), the paths along the heavy lines have equal length and the paths along the dotted lines have equal length. Furthermore, if we examine the portions of the diagram near F and L (pictured in Fig. 18), the poles show that

$$|e_1| = |e_3|, \quad |e_2| = |e_4|, \quad |e_5| = |e_7|, \quad \text{and} \quad |e_6| = |e_8|.$$

Since v'' is a geodesic, $|e_3| + |e_7| \geq |e_4| + |e_8|$. Therefore,

$$|e_1| + |e_5| \geq |e_2| + |e_6|.$$

This shows that $|\gamma| \geq |\gamma'|$. Notice also that this shows that the paths ω and λ are m-parallel. Thus, by Lemma 30 the words ω and λ are $2m$-fellow travelers. Therefore, all that is needed is to show that $\lambda \prec \omega$.

If $|\gamma| > |\gamma'|$ then it is clear that $\lambda \prec \omega$. Suppose that $|\gamma| = |\gamma'|$. First assume that $ext(F) \cap \omega''$ contains an edge. Let a be the letter of the word ω that comes just before the occurrence of γ. We will show that the word $a\gamma$ is an excessive word, with γ and γ' the high and low words. Because there is a vertex of degree three between the letter a and the word γ, the first letter of γ must be a. By the construction of γ', $\bar{\gamma} = \bar{\gamma}'$. We now need to show that the first two letters of γ' have different colors.

If the first region of Δ, W_1, is a side region of M_{th} then γ' partially labels two interior edges of that region. Because of the form of the relators the first two letters of γ' must have different colors. Suppose W_1 is a corner. Then γ' only passes over one of the interior edges of that region. Call this edge e. If e is labeled by a single letter then by the position of the poles, $ext(W_1)$ is labeled by at least half a relator. This implies that ω contains a far-corner or Dehn word, which contradicts the hypothesis. Thus, e must be labeled by a piece that is at least two letters long. Therefore, the subword $a\gamma$ of ω is an excessive word. By Corollary 6, we have that $\lambda \prec \omega$.

Now consider the case when $ext(F) \cap v''$ contains an edge. Let φ be the word labeling $ext(\nabla)$ and let φ' be the word labeling $base(\nabla)$. The word obtained from v' by replacing the occurrence of φ by the word φ' can be be shown to precede v' in the ordering. This can be shown in the same manner as was done in the above two paragraphs. But this contradicts the choice of $v' \in L(G)$. Therefore, $ext(F) \cap v''$ must not contain an edge and $ext(F) \cap \omega''$ does contain an edge. This completes the proof. $\square$

9. Refuting non-geodesics

A word $\omega \in \mathscr{A}^*$ is called a *geodesic* if it has minimal length among all words in $\mathscr{A}^*$ that represent the same group element. Equivalently, this means that the corresponding path in the Cayley graph is a geodesic. The goal in this section is to refute all non-geodesic words. The approach is similar to the approach used in Section 8 to refute the two generator non-geodesics. The main result is the following proposition.

Proposition 37. *If H is an Artin group of extra-large type, with $m = \max\{m_{ij} \mid m_{ij} \neq \infty\}$, then all words which are non-geodesics are 2m-refuted.*

To prove Proposition 37 assume that ω is a non-geodesic word in $\mathscr{A}^*$. By the results of the two previous sections we may assume that ω does not contain any inverse pairs, Dehn words, far-corners, strip words or two generator non-geodesics. Let ω' be of minimal length among all non-geodesic subwords of ω. Choose $v' \in L(H)$ such that $\bar{\omega}' = \bar{v}'$.

The presentation $\langle \mathscr{X} \mid \mathscr{R} \rangle$ does not in general satisfy the small cancellation hypothesis $C(4)$–$T(4)$. However, we will show that there is an equality $\mathscr{R}$-diagram, M, for the words v' and ω', that does satisfy $C(4)$–$T(4)$.

First construct a new presentation for H. Recall that for each distinct pair of generators, $x_i, x_j \in \mathscr{X}$, H_{ij} is the group given by the presentation $\langle x_i, x_j | r_{ij} \rangle$. For each pair (i, j), let $\mathscr{S}_{ij}$ be the set of all cyclically reduced words in $\mathscr{A}_{ij}^*$ which are equal to the identity in H_{ij}. Define $\mathscr{S} = \bigcup_{i \neq j} \mathscr{S}_{ij}$. Clearly, $\langle \mathscr{X} | \mathscr{S} \rangle$ is another presentation for the group H. Notice that the presentation $\langle \mathscr{X} | \mathscr{S} \rangle$ is infinite. In their paper [2], Appel and Schupp study this presentation. They show that for any word u, an $\mathscr{S}$-diagram can be constructed satisfying the following properties: the boundary label is u, the diagram satisfy $C(8)$, and each interior edge is labeled by one syllable. Recall that a syllable is a power of a single semigroup generator.

Let $M_{\mathscr{S}}$ be an equality $\mathscr{S}$-diagram for the words v' and ω'. Let p_0 and p_f be the point of $\partial M_{\mathscr{S}}$ where the two words ω' and v' begin and end, respectively. As noted above, we can assume that $M_{\mathscr{S}}$ satisfies $C(8)$ and each interior edge is labeled by a single syllable.

Lemma 38. *The $\mathscr{S}$-diagram $M_{\mathscr{S}}$ is a basic thin equality diagram. Furthermore, for each region D, $\partial D \cap \omega'$ and $\partial D \cap v'$ are edges which are labeled by at least two syllables. Thus, both generators appear on each of $\partial D \cap \omega'$ and $\partial D \cap v'$.*

Proof. To begin with, note that the $\mathscr{S}$-diagram, $M_{\mathscr{S}}$ satisfies $C(8)$. The property $C(8)$ implies $C(6)$, thus by Corollary 9, $\sum^*(4 - i(D)) \geq 6$.

First we will prove the result in the case the word ω' represents the identity in H. In this case, $v' \in L(H)$ is the empty word. The diagram is therefore labeled by the word ω'. If the diagram has one region, then ω' is a two generator word and is therefore a two generator non-geodesic. This contradicts the choice of ω, since we have assumed that ω does not contain two generator non-geodesics. If the diagram has more than one region, then because $\sum^*(4 - i(D)) \geq 6$, there are at least two simple boundary regions with three or fewer syllables on their bases. These regions must each have at least five syllables on their exteriors. Therefore, by Lemma 19, the bases are shorter then the exteriors. This means that ω' must have a subword which is a two generator non-geodesic, which again contradicts the choice of ω. This completes the proof in the case that ω' represents the identity element.

If $M_{\mathscr{S}}$ contains a cut vertex which is touched by ω' more than once, then ω' is not minimal. If $M_{\mathscr{S}}$ contains a cut vertex which is touched by v' more than once, then v' is not a geodesic. If $M_{\mathscr{S}}$ contains a cut vertex which touches both ω' and v' then ω' was not minimal. Therefore $M_{\mathscr{S}}$ has no cut vertices.

Notice that $M_{\mathscr{S}}$ must not contain any spikes since ω' and v' are freely reduced. If $M_{\mathscr{S}}$ begins or ends with a bridge to the point p_0 or the point p_f, this will contradict the minimality of ω'. If the diagram contains a singleton strip then the label on the boundary of the singleton strip must not be within ω' or v'. Thus, any singleton strip must occur at the beginning of $M_{\mathscr{S}}$, at p_0, or at the end of $M_{\mathscr{S}}$, at p_f. If $M_{\mathscr{S}}$ begins or ends with a singleton strip remove the singleton strip from the diagram. Continue to remove singleton strips. The removed portion of the diagram will be thin.

Now suppose that this reduction procedure terminates with a one region diagram. Then the whole $\mathscr{S}$-diagram is thin. If the procedure terminates with any other subdiagram, $M'_{\mathscr{S}}$, then $M'_{\mathscr{S}}$ satisfies $\sum^*(4 - i(D)) \geq 6$. Since $M'_{\mathscr{S}}$ has no spikes or singleton strips, any simple boundary region must have interior degree greater than or equal to two. But this implies that there are at least three simple boundary regions with interior degree two or three. One of these regions must have its entire exterior in ω' or v'. By Lemma 19, since the exterior contains at least five syllables and the base contains only two or three syllables, the interior is shorter than the exterior. This either contradicts the choice of ω, which contains no two generator non-geodesics, or contradicts the choice of v', which is a geodesics. Therefore, the reduction procedure must terminate with a one region subdiagram and the diagram is thin.

Now we will show that for each region D of $M_{\mathscr{S}}$, $\partial D \cap \omega'$ and $\partial D \cap v'$ each contain at least two syllables. Suppose, for example, that there is a region D such that $\partial D \cap v'$ does not contain two syllables. Then we can show that the other word ω' has a two generator non-geodesic subword. Notice that the labels on the interior edges of the region D are both at most one syllable. The label on $\partial D \cap \omega'$ must be at least five syllables and the label on the rest of the region D is at most three syllables. Therefore by Lemma 19, the subword of ω' which labels $\partial D \cap \omega'$ is a two generator non-geodesic. This is a contradiction, therefore $\partial D \cap v'$ must be labeled by at least a two syllable word. In a similar manner we can show that if $\partial D \cap \omega'$ must be labeled by a two or more syllable word. This proves the lemma. □

Notice that Lemma 38 is the only place in this paper that requires that the Artin group be of extra-large type, as opposed to large type.

The diagram $M_{\mathscr{S}}$ is used to construct a $C(4)$–$T(4)$ $\mathscr{R}$-diagram. Each $\mathscr{S}$-region, D, is labeled by a word equal to the identity in one of the H_{ij}. The presentations $\langle x_i, x_j \mid r_{ij} \rangle$ satisfy $C(4)$–$T(4)$. Each of the $\mathscr{S}$-region can be filled in with a $\mathscr{R}_{ij}$-diagram. Denote this "filled in" diagram by M. This diagram M is an $\mathscr{R}$-diagram, where $\mathscr{R}$ is the finite set of relators for the Artin group. In Lemma 40 we will show that M satisfies $C(4)$–$T(4)$. However, before this we need to take care of a small detail. It is possible when filling in an $\mathscr{S}$-region, D, that two points of the boundary of an $\mathscr{S}$-region are pulled together to form a *pinch*. The next lemma will show that this situation will not occur when we fill in the diagram $M_{\mathscr{S}}$.

Lemma 39. *Suppose that D is a $\mathscr{S}$-region, when filling in the region D with an $\mathscr{R}_{ij}$-diagram it is not possible to pinch together any two points of ∂D creating a cut vertex.*

Proof. Suppose that D is a $\mathscr{S}$-region. Then the label on ∂D is a cyclically reduced word in $\mathscr{A}_{ij}$ representing the identity element in the group H_{ij}. The boundary of D consists of the following boundary segments: $\partial D \cap \omega'$; $\partial D \cap v'$; the interior edge, s', shared with the $\mathscr{S}$-region just before D; and the interior edge, s'', shared with the $\mathscr{S}$-region just after D.

If two points of $\partial D \cap \omega'$ are pinched then the word ω' must contain a two generator word which is equivalent to the identity in the group H_{ij}. Thus, ω' contains a two generator non-geodesic subword. But we have assumed that ω' does not contain two generator non-geodesic subwords, so this is a contradiction. Similarly, two points of $\partial D \cap v'$ cannot be pinched. Suppose that a point of $\partial D \cap \omega'$ is pinched to a point of one of the interior edges s' or s''. This would imply that there is a two generator subword, γ, of ω' which is equivalent to a one syllable word in the group H_{ij}. By Lemma 19, the word γ is a two generator non-geodesic, again a contradiction. Similarly, a pinch between a point of $\partial D \cap v'$ and a point of an interior edge of D leads to a contradiction.

If a point of one interior edge pinches to a point of the other interior edge then the words on $\partial D \cap \omega'$ and $\partial D \cap v'$ are equivalent to words of two syllables and again by Lemma 19, ω' and v' contain two generator non-geodesics, which is a contradiction.

Finally, it is not possible for a pinch to occur between two points of the same interior edge. This would imply that there is an $\mathscr{R}_{ij}$-subdiagram labeled by one syllable. But by Lemma 19, there are no words equal to the identity which have less than $2m_{ij}$ syllables, except the empty word. □

Lemma 40. *The $\mathscr{R}$-diagram, M, satisfies $C(4)$-$T(4)$.*

Proof. First we show that M is $T(4)$. Any interior vertex of M which was an interior vertex of one of the $\mathscr{R}_{ij}$-diagrams will have degree four since each of the H_{ij} satisfy $T(4)$. Suppose v is an interior vertex of M which is not an interior vertex of one of the $\mathscr{R}_{ij}$-diagrams. The vertex v must lie along one of the interior edges of the $\mathscr{S}$-diagram. Since each interior edge of the $\mathscr{S}$-diagram is labeled by a power of a single semigroup generator, as a boundary vertex of a $\mathscr{R}_{ij}$-diagram the vertex v must have at least degree three (no relator has two consecutive generators). This is true for both of the $\mathscr{R}_{ij}$-diagrams which border the interior edge of the $\mathscr{S}$-diagram. Therefore, the degree of v in the $\mathscr{R}$-diagram is at least four. This shows that M satisfies $T(4)$.

Let D be an almost interior region of M. If D is an almost interior region of one of the $\mathscr{R}_{ij}$-diagrams then the interior degree of D is at least four in the $\mathscr{R}_{ij}$-diagram and is therefore at least four in the diagram M. Suppose D was a boundary region of an $\mathscr{R}_{ij}$-diagram. Then since D has no edges on the boundary of the $\mathscr{R}$-diagram, M, the exterior of D relative to the $\mathscr{R}_{ij}$-diagram must lie along one of the interior edges of the $\mathscr{S}$-diagram, $M_{\mathscr{S}}$. Thus, in the $\mathscr{R}_{ij}$-diagram, the exterior of D is a single edge labeled by a single letter. By the form of the relators, the base of D in the $\mathscr{R}_{ij}$-diagram must consist of at least three edges. Therefore, in the $\mathscr{R}$-diagram, M, the interior degree of D is at least four. □

If the $\mathscr{R}$-diagram M is thin then by Proposition 21, there is a word that $2m$-refutes ω. Therefore, we concentrate on the case when M is not a thin equality diagram. Assume that M is not thin. Notice that the two boundary words do not label strips on spikes. Therefore, M is a diagram like those discussed in Section 6. Let M_{th} be the first thick section of M. Let ω'' be the label on $\omega' \cap \partial M_{\text{th}}$ and v'' be the label on $v' \cap \partial M_{\text{th}}$. Let F and L be the first and last regions of M_{th}. Let the sequences $\Delta = \{W_1, W_2, \dots, W_s\}$ and $\nabla = \{V_1, V_2, \dots, V_r\}$ be the sequences of boundary regions along the top and bottom of M_{th}. The diagram M_{th} will look like the diagram in Fig. 5.

It is clear by Lemma 38 that the diagram M_{th} is a sequence of $\mathscr{R}_{ij}$-subdiagrams, where each subdiagram corresponds to one of the $\mathscr{S}$-regions of the original diagram $M_{\mathscr{S}}$. Let $\{M_0, M_1, \dots, M_n\}$ be the sequence of $\mathscr{R}_{ij}$-subdiagrams of M_{th}. The sequences Δ and ∇ can be divided into subsequences $\{\Delta_0, \Delta_1, \dots, \Delta_n\}$ and $\{\nabla_0, \nabla_1, \dots, \nabla_n\}$ where each of the subsequences Δ_i and ∇_i consist of regions from the subdiagram M_i.

Lemma 41. *Suppose $\alpha = \partial M_i \cap \partial M_{i+1}$ for some $0 \le i \le n-1$. Suppose D is an $\mathscr{R}$-region of the diagram M_{th}. Then $\partial D \cap \alpha$ is a consecutive portion of ∂D. Furthermore, $\partial D \cap \alpha$ is no more than a single edge labeled by a single letter.*

Proof. Suppose that an $\mathscr{R}$-region D touches α at two or more non-consecutive portions of ∂D. Then there will be a subdiagram K, of regions labeled by the same relator as D, bounded by the region D and α. The boundary of K is contained in ∂D and α. The subdiagram $D \cup K$ is an $\mathscr{R}_{ij}$-diagram and contains more than one region. By Corollary 11, there are at least two disjoint strips in the diagram $D \cup K$. This implies that the label on α contains a strip word. But α is labeled by a single syllable and cannot contain a strip word. Therefore, all $\mathscr{R}$-regions touch α in a consecutive portion of their boundary.

Since the syllables of all relators are one letter long, it is clear that $\partial D \cap \alpha$ is no more than a single edge labeled by a single letter. □

Most of the remaining argument will focus on the structure of the first $\mathscr{R}_{ij}$-subdiagram M_0. Suppose that m_{ij} is the half length of the relator labeling the regions of M_0. Consider the sequences $\Delta_0 = \{W_1, W_2, \dots, W_k\}$ and $\nabla_0 = \{V_1, V_2, \dots, V_l\}$. Let α_0 be the $\partial M_0 \cap \partial M_1$. The next three lemmas state simple facts about the boundary regions of M_0.

Lemma 42. *The regions W_k and V_l each have a boundary edge in α_0. These regions are simple boundary regions of M_0, $i(W_k, M_{\text{th}}) = i(W_k, M_0) + 1$, and $i(V_l, M_{\text{th}}) = i(V_l, M_0) + 1$.*

Lemma 43. *Any regions of Δ_0 and ∇_0, other than W_k and V_l, that touch α_0 are non-simple boundary regions of M_0.*

Lemma 44. *Simple boundary regions of M_0 are either simple boundary regions of M_{th} or interior regions of M_{th}.*

There may be a set of simple boundary regions of M_0 which are interior regions of M_{th}. These are a subset of the regions of M_0 which touch α_0. Relative to M_0, these regions all have interior degree three and have exterior labeled by a single letter. Denote this set of regions by Φ.

Lemma 45. *Either* $\tau(\Delta_0, M_{th}) = 1$ *or* $\tau(\nabla_0, M_{th}) = 1$.

Proof. To prove this lemma we will examine the strips of the diagram M_0.

If W_k is a singleton strip of M_0 then it is a corner in the diagram M_{th}. This implies that $\tau(\Delta_0, M_{th}) = 1$. Similarly if V_l is a singleton strip of M_0 then $\tau(\nabla_0, M_{th}) = 1$. If a compound strip of M_0 begins in Δ_0 and ends with the region W_k, then W_k is a side region in the diagram M_{th} and $\tau(\Delta_0, M_{th}) = 1$. Similarly, if a compound strip of M_0 begins in ∇_0 and ends with the region V_l, then the region V_l is a side region of M_{th} and $\tau(\nabla_0, M_{th}) = 1$.

The only regions of M_0 which can be singleton strips are the regions F, W_k, and V_l. If any other region of Δ_0 or ∇_0 were a singleton strip then there would be a Dehn word in ω' or v'. And the regions of Φ all have interior degree three in M_0 so cannot be singleton strips. In the above paragraph we have proved the lemma in the case when W_k or V_l is a singleton strip. Therefore, assume that W_k and V_l are not singleton strips of M_0.

Suppose that F is a singleton strip of M_0. By Corollary 11 there are at least two compound strips of M_0. If one of the compound strips is made up of regions from Δ_0, then the compound strip must end at the region W_k or else the word ω' would contain a strip word. As in the first paragraph of this proof, this implies that $\tau(\Delta_0, M_{th}) = 1$. Similarly, a compound strip made up of regions from ∇_0 implies $\tau(\nabla_0, M_{th}) = 1$. Thus, we may assume that at least one strip ends with a region from Φ. But the regions of Φ are not corners of M_0, they all have interior degree three. So F cannot be a singleton strip.

Suppose that $i(F, M_0) > 1$. Then by Corollary 11 there are four compound strips of M_0. Again an argument similar to that in the preceding paragraph will lead to a contradiction. This proves the lemma. □

Proof of Proposition 37. If the diagram M is thin then by Proposition 21 there is a word that $2m$-refutes ω. Suppose that M is not thin. We will consider four distinct cases, that exhaust all possibilities.

Case 1: $\partial F \cap \omega'$ includes an edge and $\tau(\Delta_0, M_{th}) = 1$.

By the methods used in Lemma 35, we can show that every region of Δ_0 has degree four and the poles for each region lie at the north and south vertices.

As was done in Section 8, the poles can be used to show the following: The label from the vertex r_1 to the vertex r_3, along $ext(\Delta_0, M_0)$, has the same length as the label from the vertex r_2 to the vertex r_4, along $base(\Delta_0, M_0)$. The edge e_4 is labeled by a single letter. Thus, the edge e_3 is labeled by a word of length $m_{ij} - 1$. Because ω' does not contain two generator non-geodesics, $|e_1| = 1$ and $|e_2| = m_{ij} - 1$. (The notation $|e_i|$ denotes the length of the label on the edge e_i.)

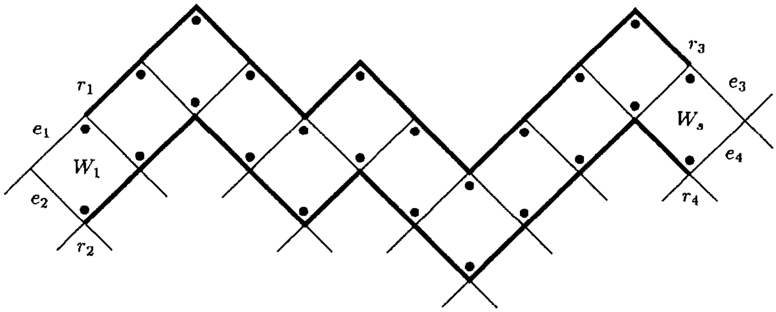

Fig. 19. The regions of Δ_0 of M_0 in case 1.

Let γ be the label on $ext(\Delta_0, M_0)$ and let γ' be the label on $base(\Delta_0, M_0)$. Let ρ be the word obtained from ω by replacing the occurrence of γ in ω by γ'. By examining the diagram in Fig. 19 it is easy to see that the paths ω and ρ are m_{ij}-parallel. Therefore, by Lemma 30, the paths ω and ρ are $2m_{ij}$-fellow travelers.

The first two letters of γ' have different colors, because the γ' starts by labeling an edge of length $m_{ij}-1$ and we are assuming that each m_{ij} is four or more. Thus, it is easy to see that ω contains an excessive word with γ and γ' the associated high and low words. Therefore, by Corollary 6, $\rho \prec \omega$ and thus ρ $2m_{ij}$-refutes ω.

Case 2: $\partial F \cap v'$ includes an edge and $\tau(V_0, M_{th}) = 1$.

Just as in Case 1, we can find an excessive subword of v (and a word ρ which $2m_{ij}$-refutes v). But by Lemma 5, this is not possible because $v \in L(G)$.

Case 3: $\partial F \cap \omega'$ includes an edge and $\tau(V_0, M_{th}) = 1$.

Notice that we may assume that $\partial F \cap v'$ does not include an edge; otherwise the result follows from Case 2. The following proof depends on the location of the first corner region of Δ.

Suppose that the first corner of Δ occurs in M_0. Then by Lemma 34 the poles for each region of Δ, up to and including the corner, are north and south. Furthermore, one of the poles for the region F is at the vertex v_1 of Fig. 20. If the other pole is within the edge e_5, then the word ω' would contain a strip word. We have assumed this is not the case.Thus, the other pole for the region F must be at the vertex v_2. By Lemma 14, v_3 is a pole for the region V_1.

Now by using a similar argument as in the proof of Lemma 35, we can show that every region of V_0 has degree four and the poles for each region lie at the north and south vertices. Because $|e_6| = 1$, we have $|e_7| = m_{ij} - 1$. Since v' is a geodesic we have

$$|e_3| = m_{ij} - 1 \quad \text{and} \quad |e_4| = 1.$$

By the positioning of the poles we have

$$|e_1| = m_{ij} - 1 \quad \text{and} \quad |e_2| = 1.$$

Let $\Omega = \{W_1, W_2, \ldots, W_h\}$ be the initial subsequence of regions of Δ_0 up to W_h the first corner. If $h = 1$, then the first region of Δ is a corner. Since $|e_2| = 1$ and

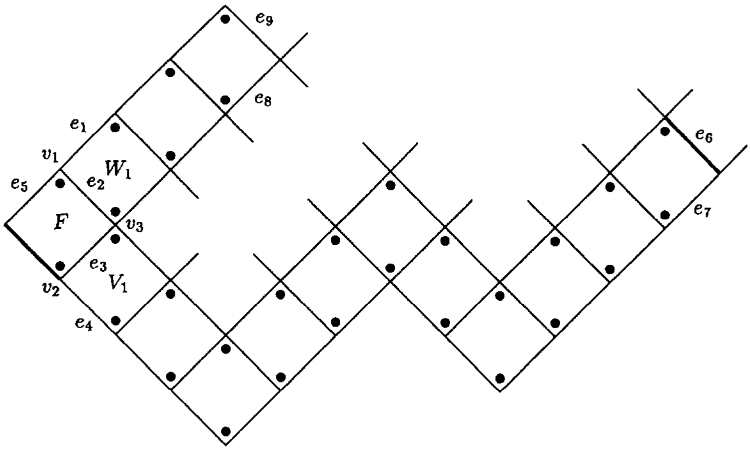

Fig. 20. The regions in Case 3.

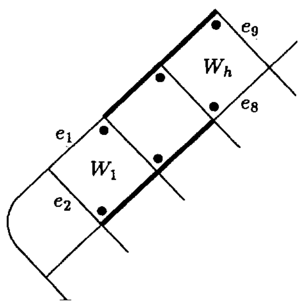

Fig. 21. The sequence Ω.

$\partial F \cap \omega'$ includes an edge, the word ω' must contain a far-corner. But we have assumed this is not the case.

If $h > 1$, then consider Fig. 21. Let λ be the label on $ext(\Omega)$ and let λ' be the label on $base(\Omega)$. Let ρ be the word obtained by replacing the occurrence of λ in ω by λ'. The dark edges in Fig. 21 will have the same length labels. Notice that $|e_8| = m_{ij} - 1$ and $|e_9| = 1$, because any other values would imply that ω' is a two generator non-geodesic, which is a contradiction. From the figure it is easy to see that Lemma 30 applies and thus ρ is a $2m_{ij}$-fellow traveler of ω. Furthermore, since $\partial F \cap \omega'$ includes an edge, it is easy to see that ω' contains an excessive word with associated high and low words λ and λ'. Thus by Corollary 6, $\rho \prec \omega$. Thus, the word ρ $2m_{ij}$-refutes ω.

Now consider the case where the first corner of Δ is not in the subdiagram M_0. Suppose that the first corner is in the subdiagram M_l. Let $\Lambda = \{W_n, W_{n+1}, \dots, W_{n+o}\}$ be the initial subsequence of Δ_l up to and including the first corner.

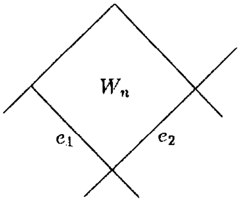

Fig. 22. Λ One region.

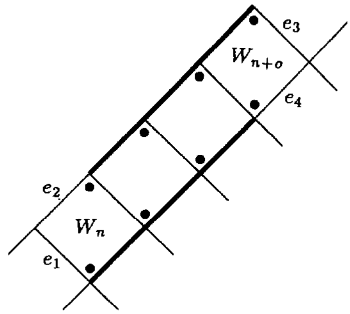

Fig. 23. Λ more than one region.

Suppose that Λ is one region long; see Fig. 22. The edge e_1 must be labeled by a single letter, and the edge e_2 is labeled by a word of length $m_{ij} - 1$ or less, where m_{ij} corresponds to the relator labeling the regions of Δ_l. Thus, the $ext(W_n, M_{\text{th}})$ is labeled by at least half a relator. We have assumed that ω does not contain any Dehn words, thus $ext(W_n, M_{\text{th}})$ must be labeled by a half relator. Let M_{l-1} be the subdiagram which precedes M_l in the diagram M_{th}. By Lemma 38, $\partial M_{l-1} \cap \omega'$ is labeled by at least two syllables. Thus it is easy to check that the word ω' contains a type two far corner, where the associated high word labels $ext(W_{n+o}, M_{\text{th}})$. But we have assumed that ω does not contain far-corners. Therefore, Λ must be longer than one region.

Suppose Λ is longer than one region; see Fig. 23. The poles of the last region W_{n+o} must be north and south or ω' would contain a far-corner. Now we can work our way back down the sequence of regions, as was done in Lemma 34, to show that all the regions of Λ have north and south poles. The edge e_1 is labeled by a single letter, thus

$$|e_1| = 1 \quad \text{and} \quad |e_2| = m_{ij} - 1.$$

Now by noticing that the dark edges in Fig. 23 have the same length, since we have assumed that ω does not contain a two generator non-geodesic,

$$|e_3| = 1 \quad \text{and} \quad |e_4| = m_{ij} - 1.$$

Therefore, the label on $ext(\Lambda, M_{\text{th}})$ is the same length as the label on $base(\Lambda, M_{\text{th}})$. Let λ be the label on $ext(\Lambda, M_{\text{th}})$, and let λ' be the label on $base(\Lambda, M_{\text{th}})$. Let ρ be the word obtained by replacing the occurrence of λ in ω by the word λ'. By the Lemma 30 ω and ρ are $2m_{ij}$-fellow travelers. By examining the diagram it is easy to see that ω contains an excessive word, with high and low words λ and λ', respectively. By Corollary 6, $\rho \prec \omega$. Therefore ρ $2m_{ij}$-refutes ω.

Case 4: $\partial F \cap v'$ includes an edge and $\tau(\Delta_0, M_{\text{th}}) = 1$.

The same argument as in Case 3, will produce a word which $2m_{ij}$-refutes v. But by Lemma 5, this is not possible because $v \in L(G)$. This completes the proof. □

10. $L(G)$ is a regular bicombing

At this point we have the necessary pieces to prove the main result of the paper, Theorem A, which states that Artin groups of extra-large type are biautomatic. By Proposition 2, it suffices to show that the group language $L(H)$ is regular and is a bicombing of the group H. Proposition 46 states that $L(H)$ is regular and Proposition 47 states that $L(H)$ is a bicombing of H.

Proposition 46. *For an Artin group of extra-large type H, the language $L(H)$, defined in Section 2, is a regular.*

Proof. Let $m = \max\{m_{ij} | m_{ij} \neq \infty\}$. We will show that each word $\omega \in \overline{L(H)}$ is $2m$-refuted. In Propositions 32, 33 and 37, we have covered the cases for which ω contains an inverse pair, Dehn word, far-corner, strip word, or a non-geodesic. Therefore we will assume that ω does not contain any of these types of subwords. Take $v \in \mathrm{L(G)}$ such that $\bar{v} = \bar{\omega}$. Construct an equality $\mathscr{R}$-diagram, M, for the words v and ω in the same way as was done in Section 9.

If M is not thin then there is a first thick subdiagram M_{th}. If all the regions of M_{th} are labeled by the same relator, then the results of Section 8 show that there is a word which $2m$-refutes ω. If the regions of M_{th} are labeled by different relators then the results of Section 9 show that there is a word which $2m$-refutes ω.

On the other hand, suppose the diagram M is thin. Because both ω and v are geodesics, v is a $2m$-fellow traveler of ω, by Lemma 21. Thus v $2m$-refutes ω. □

Proposition 47. *For an Artin group of extra-large type H, the language $L(H)$, defined in Section 2, is a bicombing of H.*

Proof. Let $m = \max\{m_{ij} | m_{ij} \neq \infty\}$. Recall the definition of a bicombing of H. For any words $v, \omega \in L(H)$ and $a, b \in \mathscr{A} \cup \emptyset$ with $\overline{\omega a} = \overline{bv}$, in the Cayley graph $\Gamma_{\mathscr{A}}(H)$ the path ω, starting at the origin, and the path v, starting at the vertex corresponding to the group element $\bar{b}$, are $(2m + 1)$-fellow travelers.

Suppose that v and ω are elements of $L(H)$, and a and b are in $\mathscr{A}$. Construct an equality $\mathscr{R}$-diagram M for the words ωa and bv as was done in Section 9. M will contain no spikes when considered as an equality diagram for at least one of the following pairs $(\omega a, bv)$, $(b^{-1}\omega a, v)$, $(b^{-1}\omega, va^{-1})$, (ω, bva^{-1}).

Suppose that M is not a thin equality diagram. Then there is a subdiagram M_{th} as described in Section 6. If all the regions of M_{th} are labeled by the same relator then the results of Section 8 imply that $\omega \notin L(G)$ or $v \notin L(G)$, which is a contradiction. If the regions of M_{th} are labeled by different relators then the results of Section 9 imply that $\omega \notin L(G)$ or $v \notin L(G)$, which is again a contradiction. Therefore M is thin.

Because the equality diagram is thin and the maximum length of a relator is $2m$, by Lemma 20 ω and v lie in a m-Hausdorff neighborhood of each other. Since ω and

v begin at most one unit apart and are geodesics, Proposition 1 shows that ω and v are $(2m + 1)$-fellow travelers. □

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